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Short time evolution of statistical characteristics for partially coherent waves

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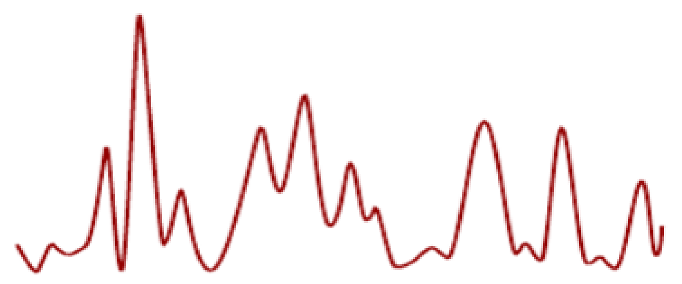
EPSRC Engineering and Physical Sciences Research Council

Integrable Turbulence



random initial condition $\psi_0(x)$

+

±1 focusing/defocusing

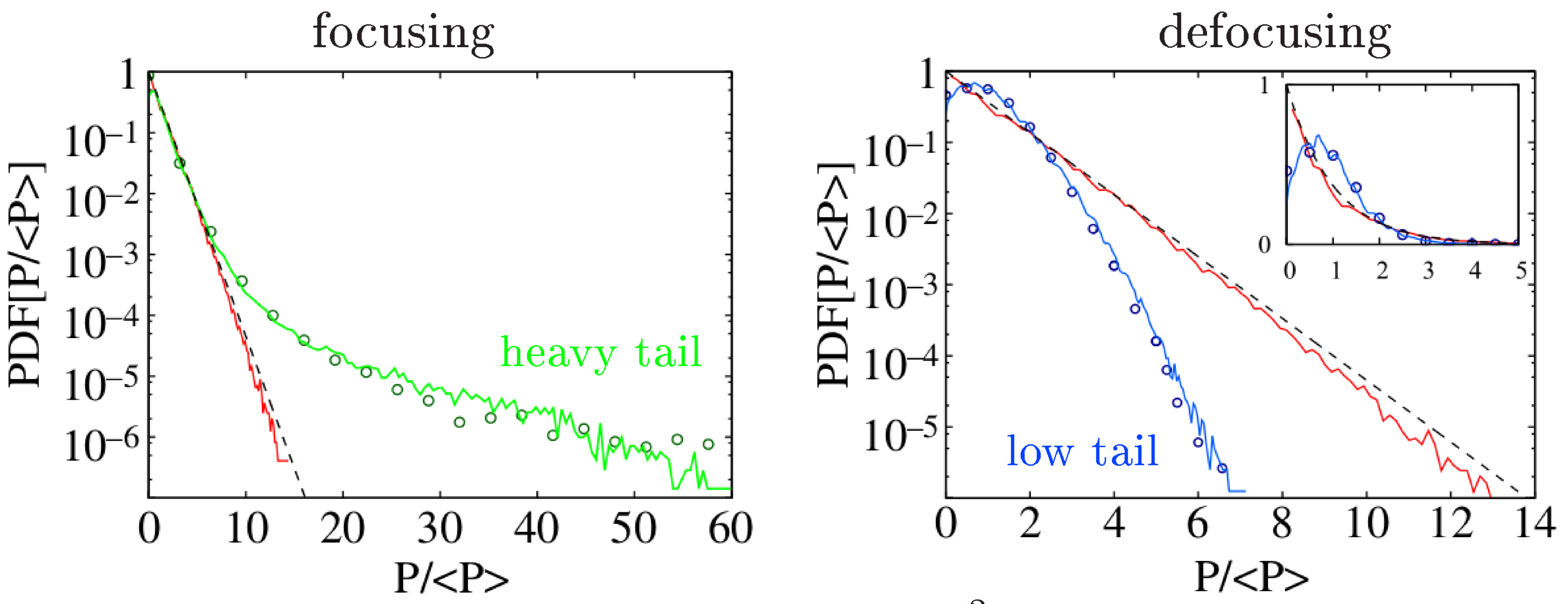
$$i\varepsilon\frac{\partial\psi}{\partial t} + \frac{\varepsilon^2}{2}\frac{\partial^2\psi}{\partial x^2} + \sigma|\psi|^2\psi = 0$$

Nonlinear Schrödinger Equation (integrable equation)

→

Given a randomly distributed initial signal we are interested in the time evolution of its statistical characteristics. In particular, we will focus on the Probability Density Function (PDF), and the Normalized Fourth order Moment (k_4). These characteristics are a key element in the study of rogue waves formation.

Normalized Fourth order Moment



[2] PDF of normalized optical power P= $|\psi|^2$ plotted in logarithmic scale. Initial condition (red), output (green and blue) and numerics (black circle)

$\langle \dots \rangle$ → ensemble averaging

- Kurtosis ⇒ $\boxed{k_4 = \frac{\langle|\psi|^4\rangle}{\langle|\psi|^2\rangle^2} = \frac{\langle\rho^2\rangle}{\langle\rho\rangle^2}}$, Normal Distribution: $k_4 = 2$
- Madelung Transform ⇒ $\psi = \sqrt{\rho}e^{i\frac{\phi}{\varepsilon}}$ $\quad u = \frac{\partial\phi}{\partial x}$

prior to gradient catastrophe

$\varepsilon \ll 1$ ⇒ dispersionless limit

$$\text{NLS} \Rightarrow \begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + uu_x - \sigma\rho_x - \cancel{\frac{\varepsilon^2}{8}\left(\frac{1}{\rho}\frac{\partial^2\rho}{\partial x^2} + \left[\frac{1}{\rho}\frac{\partial\rho}{\partial x}\right]_x\right)_x} = 0 \end{cases}$$

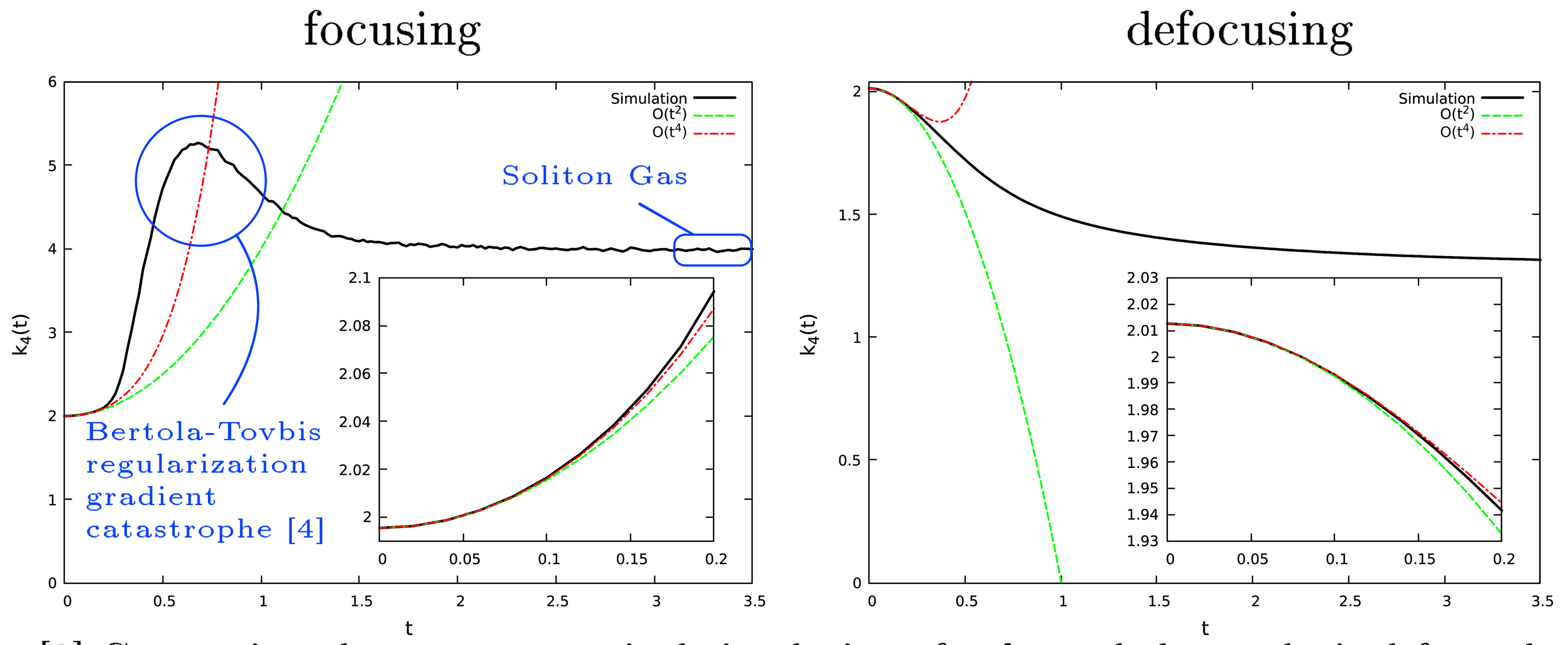
- We solve the dispersionless (hydrodynamic) system:

smooth initial contition: $u_0(x) \simeq 0$, $\quad \rho_0(x)$ random process

short time expansion: $\rho(x,t) = \sum \rho_n(x)t^n$, $u(x,t) = \sum u_n(x)t^n$, $t \ll 1$

$$\Rightarrow k_4(t) \simeq k_4(0) + \frac{\sigma}{\langle\rho_0\rangle^2}\langle\rho_0\rho_{0x}^2\rangle t^2 + - \frac{1}{2\langle\rho_0\rangle^2}\langle\frac{2}{3}\rho_0^2\rho_{0x}\rho_{0xxx} + \frac{17}{6}\rho_0\rho_{0x}^2\rho_{0xx} + \frac{1}{2}\rho_{0x}^4\rangle t^4$$

- evolution in terms of ρ_0 and its derivatives
- dominant term $\sim t^2$
- σ determines the growth/decay of $k_4(t)$
- evolution independent from ε



[3] Comparison between numerical simulations for k_4 and the analytical formula.

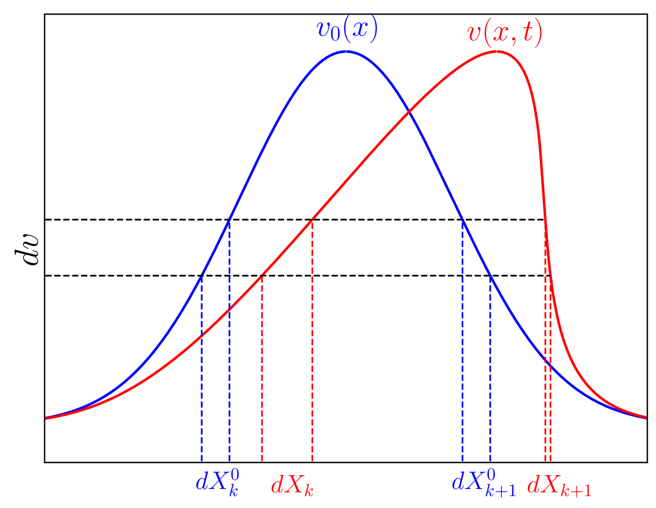
PDF

- Toy Model [5]: **Hopf equation** → $\boxed{\text{PDF conserved}}$ (before breaking)

$v_t + vv_x = 0$ with $v(x,0) = v_0(x)$ random process

solution in the characteristic form:

$x = X_k(v,t) = v_{0,k}^{-1}(v) + vt \quad (v_0^{-1}$ multi-valued)

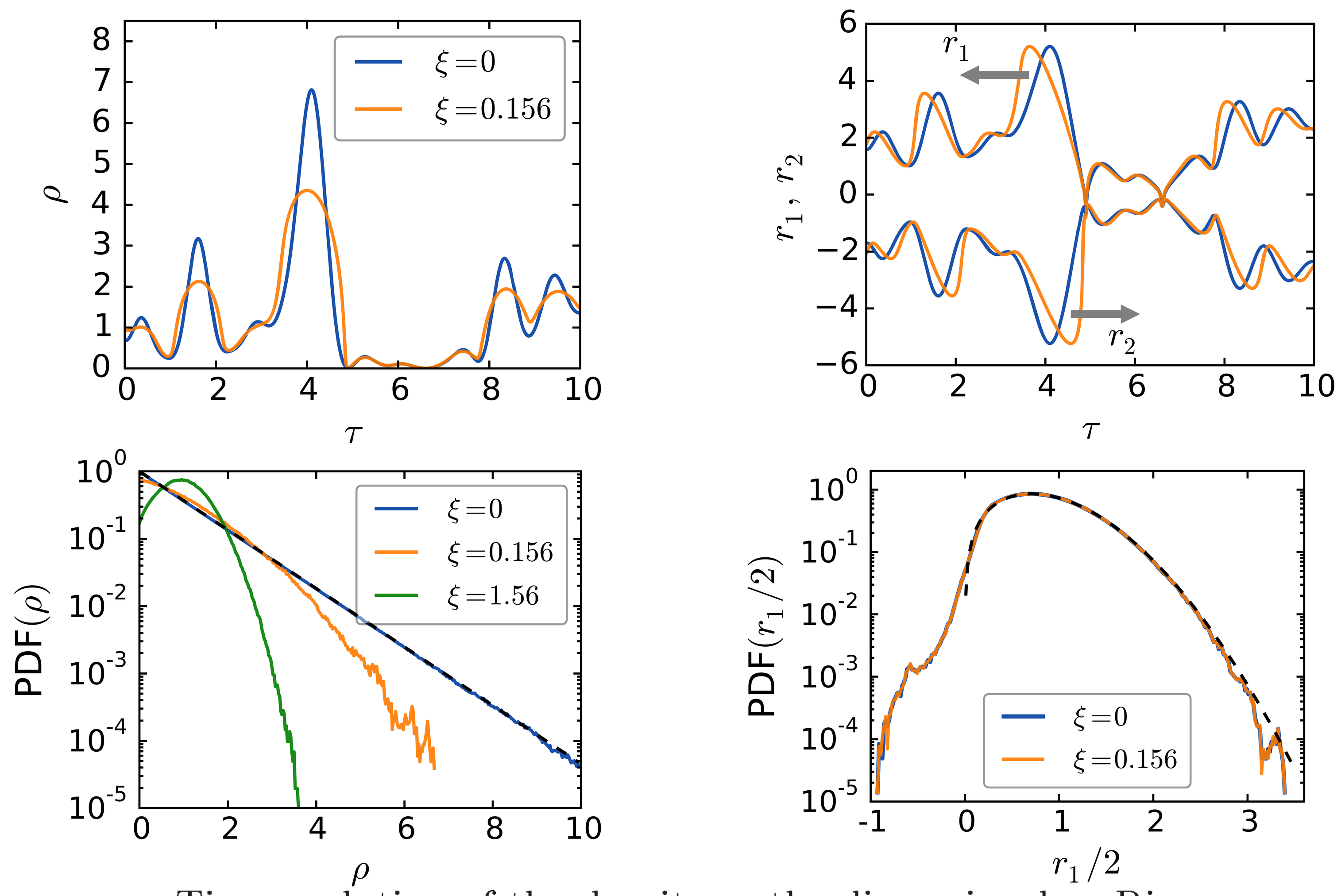


$$\frac{\partial X(v,t)}{\partial v} = \frac{1}{v_0'(v_0(v))} + t$$
$$\Downarrow$$
$$dX_k(t) = \frac{\partial X}{\partial v}dv = dX_k^0 + tdv$$
$$dX_{k+1}(t) = \frac{\partial X}{\partial v}dv = dX_{k+1}^0 - tdv$$
$$\Downarrow$$
$$dX_k(t) + dX_{k+1}(t) = dX_k^0 + dX_{k+1}^0$$

$$\mathcal{P}(v;t) = \lim_{L\to\infty}\frac{1}{L\,dv}\sum_k dX_k(t) = \lim_{L\to\infty}\frac{1}{L\,dv}\sum_k dX_k^0 = \mathcal{P}(v;0)$$

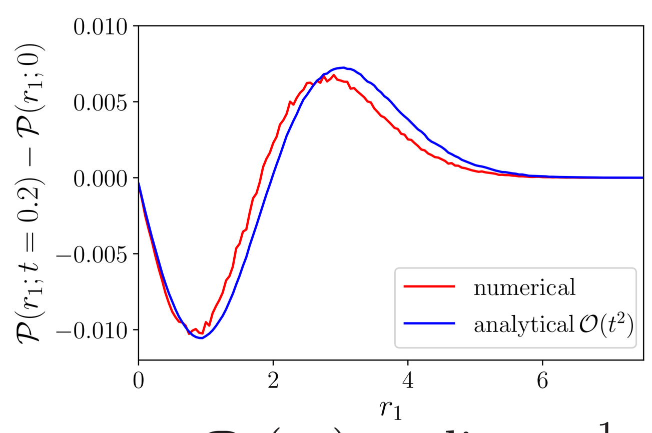
$\mathcal{P}(v;t)dv$ is the probability to have $v(x,t) \in [v - dv/2; v + dv/2]$.

- **Defocusing NLS ($\sigma = -1$)**[6]:



Time evolution of the density ρ, the dispersionless Riemann invariants $r_{1,2}$ and their corresponding PDF. Optical variables.

☞ we compute the PDF in the dispersionless limit



- **PDF of Riemann Invariant:**

$r_{1,2} = u \pm 2\sqrt{\rho}$

$r_{1,2t} + V_{1,2}(r_1,r_2)r_{1,2x} = 0$

Small time expansion (cf. Hopf example)

$\mathcal{P}(r_1;t) \simeq \mathcal{P}_0(r_1) + t^2\mathcal{P}_2(r_1)$

$$\mathcal{P}_2(r_1) = \lim_{L\to\infty}\frac{1}{L\,dr_1}\sum_k -\frac{1}{8}\left({r'_{0,k}}^2 + r_1 r''_{0,k}\right)dX_k^0,$$

- **PDF of ρ** (ongoing work):

The density is not a Riemann invariant and the small time expansion of its PDF requires a specific treatment. A first approximation yields:

$$\mathcal{P}(\rho;t) \simeq \lim_{L\to\infty}\frac{1}{L\,\rho}\sum_k\left[\left(1 - \frac{\rho\rho_{0,k}''^2 t^2}{\rho_{0,k}'^2}\right)^{-1/2} - \left(\frac{\rho\rho_{0,k}'''}{2\rho_{0,k}'} - \rho_{0,k}''\right)t^2\right]dX_k^0$$

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