Dark matter haloes may form via nearly-thermal turbulent cascades

Jonathan Skipp, University of Warwick
Sergey Nazarenko, InPhyNi – CNRS, Nice
Victor L'vov, Weizmann Institute of Science, Israel

Fuzzy dark matter and the Schrödinger-Newton Equations

Dark matter holds together large astrophysical objects like galaxies and galactic clusters today. **Dark matter haloes** are gravitationally-bound clumps of dark matter that seeded the collapse of baryonic matter into these large structures in the early Universe.

The nature of dark matter is notoriously mysterious. One candidate is **fuzzy dark matter**: self-gravitating, ultra-light bosons, which can be modelled by the **Schrödinger-Newton system of equations (SNE),**

NLS eq for bosons $\quad i\partial_t\psi + \nabla^2\psi - \psi V = 0 \qquad (1)$
Poisson eq for Gravity $\quad \nabla^2 V = Gm|\psi|^2. \qquad (2)$

The cubic NLS (eq (1) with $V = |\psi|^2$) exhibits waves, particle/energy cascades, condensates and turbulence. We conjecture that the SNE has similar phenomenology. We therefore investigate the weakly-nonlinear limit of the SNE, to examine how dark matter haloes could build up from a background of incoherent fluctuations.

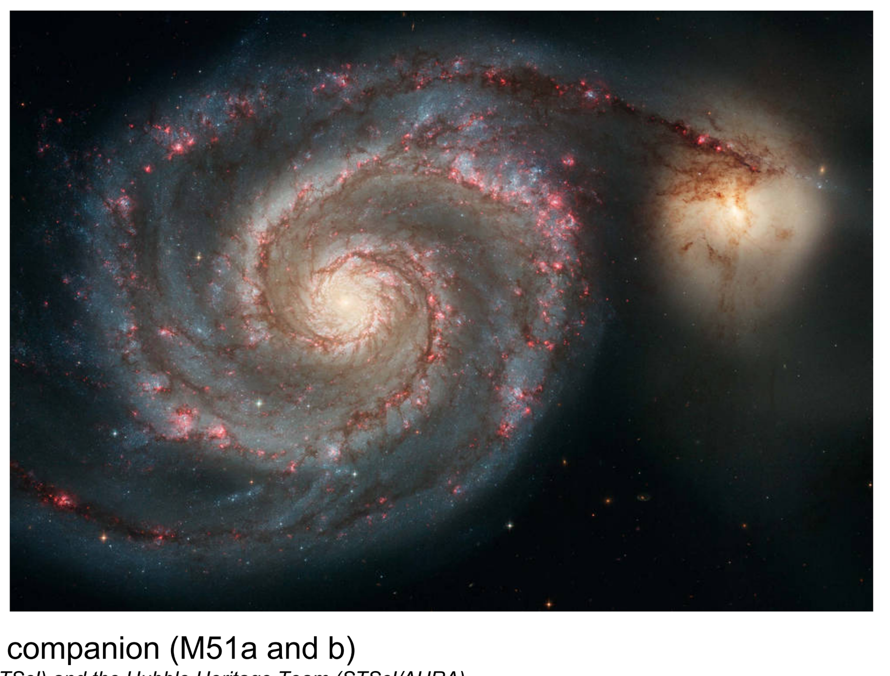

Fig 1. Whirlpool galaxy and companion (M51a and b)
Image credits: NASA, ESA, S. Beckwith (STScI) and the Hubble Heritage Team (STScI/AURA)

Wave turbulence of the SNE

Wave Turbulence studies large ensembles of weakly-nonlinear waves in Fourier space ($\psi(\mathbf{x},t) \to \hat{\psi}_\mathbf{k}(t)$). In this limit we can derive a **wave kinetic equation** which describes the time evolution of the wave spectrum $n_\mathbf{k} \sim \langle \hat{\psi}_\mathbf{k}\hat{\psi}^*_\mathbf{k}\rangle$,

$$\partial_t n_\mathbf{k} = 4\pi \int \left|W^{12}_{3\mathbf{k}}\right|^2 \delta^{12}_{3\mathbf{k}}\delta\left(\omega^{12}_{3\mathbf{k}}\right) n_1 n_2 n_3 n_\mathbf{k}\left[\frac{1}{n_1}+\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_\mathbf{k}}\right] d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \qquad (3)$$

Dispersion relation $\quad \omega_\mathbf{k} = k^2$

Resonance conditions $\quad \delta^{12}_{34} = \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4) \quad , \quad \delta(\omega^{12}_{34}) = \delta(\omega_1+\omega_2-\omega_3-\omega_4)$
4-wave interaction coeff $\quad W^{12}_{34} = \frac{Gm}{4}(A_{1234}+A_{2134}+A_{1243}+A_{2143}) \quad , \quad A_{1234} = \frac{1}{(\mathbf{k}_1-\mathbf{k}_4)\cdot(\mathbf{k}_3-\mathbf{k}_2)}$

Eq (3) has two **adiabatic invariants** that are preserved as the spectrum evolves: (4)

Particles	$N = \int n_\mathbf{k}\, d\mathbf{k}$	**particle flux** η through $\mathbf{k}$-space
Energy	$E = \int \omega_\mathbf{k} n_\mathbf{k}\, d\mathbf{k}$	**energy flux** ϵ

and stationary spectra:
Rayleigh-Jeans (RJ) – thermal equilibrium (equipartition) $n^{RJ}_\mathbf{k} = \frac{T}{\mu+\omega_\mathbf{k}}$
Equipartition of particles $\quad n^{TN}_\mathbf{k} \propto k^0$
Equipartition of energy $\quad n^{TE}_\mathbf{k} \propto k^{-2}$

Kolmogorov-Zakharov (KZ) – cascade of invariants
Particle cascade $\quad n^{FN}_\mathbf{k} \propto k^{-1}$
Energy cascade $\quad n^{FE}_\mathbf{k} \propto k^{-5/3}$

Fjørtoft argument for cascade directions

Consider the system forced at scale ω_f (E injected at rate ϵ, N injected at rate η) and dissipated at widely separated scales (**wide inertial range**) in steady state

$$\omega_{\min} \ll \omega_f \ll \omega_{\max}$$
Large-scale dissipation — Forcing — Small-scale dissipation

The factor of ω in (4) means we must have $\epsilon \sim \omega\eta$ at all scales, particularly $\epsilon \sim \omega_f\eta$.

If E is dissipated at $\omega_{\min}$ at rate ϵ then N must be dissipated at rate $\epsilon/\omega_{\min} \sim \eta\, \omega_f/\omega_{\min} \gg \eta$ i.e. greater than the injection rate. Therefore E must be mainly dissipated at $\omega_{\max}$. Similarly N must be mainly dissipated at $\omega_{\min}$.

This strongly constrains the flux directions:
$N \to$ **large scales** ($\eta < 0$), while $E \to$ **small scales** ($\epsilon > 0$).

KZ spectra predict the "wrong" cascade directions

How do the fluxes η and ϵ behave on power-law spectra $n_\mathbf{k} \sim k^{-x}$?
Steep spectra: $\epsilon, \eta > 0$ for $x \gg 1$, $\epsilon, \eta < 0$ for $x \ll 0$ (fluxes flatten steep spectra).
KZ spectra: $\epsilon = 0$ on $n^{FN}_\mathbf{k}$, $\eta = 0$ on $n^{FE}_\mathbf{k}$ (spectra of pure N and E flux respectively).
RJ spectra: $\epsilon = \eta = 0$ on $n^{TN}_\mathbf{k}$ and $n^{TE}_\mathbf{k}$ (no flux in equilibrium).
In between, continuity implies the fluxes have the signs as shown in Fig 2.

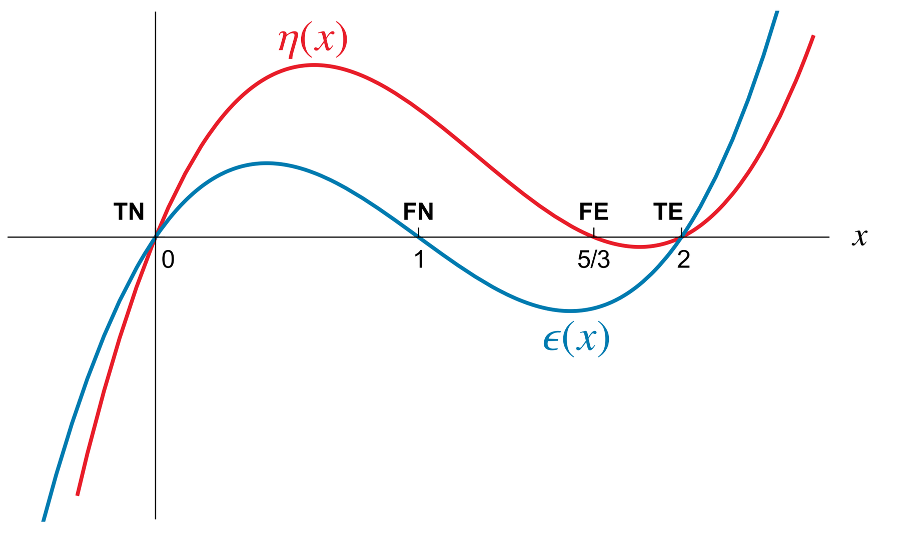

Fig 2 Particle flux η and energy flux ϵ as a function of spectral index x (i.e. $n_\mathbf{k} \sim k^{-x}$). On FN spectrum $\eta > 0$ and on FE spectrum $\epsilon < 0$, contradicting the Fjørtoft argument.

The fluxes are in the wrong direction c.f. Fjørtoft argument for both KZ spectra.
Therefore **KZ spectra cannot match any finite inertial range.**

Differential approximation model predicts warm cascades

As the KZ spectra cannot be realised, the flux must be carried by a steady-state spectrum close to the thermal RJ spectrum. This can be seen in a simplified model that has the same qualitative behaviour as eq (3). This is the **differential approximation model (DAM)**, derived by assuming $\omega_\mathbf{k} \approx \omega_1 \approx \omega_2 \approx \omega_3$:

$$\partial_t(\omega^{1/2}n) = \partial_{\omega\omega}R \quad , \quad R = \omega^{9/2}n^4\partial_{\omega\omega}(1/n)$$

Within the DAM, the fluxes are

$$\eta = -\partial_\omega R \quad , \quad \epsilon = R - \omega\partial_\omega R \qquad (5)$$

Putting a weakly-perturbed thermal spectrum $n = \frac{T}{\mu+\omega+\theta(\omega)}$ into eqs (5) predicts

$$\eta = -\frac{15\omega_{\min}^{3/2}}{4}\left(\frac{T}{\mu}\right)^3 \quad , \quad \epsilon = \frac{3T^3}{4\omega_{\max}^{1/2}}$$

i.e. the **thermodynamic potentials are set by the fluxes and dissipation scales**.

The result is a **warm cascade** – a nearly-thermal spectrum that carries the flux from the forcing to the dissipation scales. This is shown in Fig 3.

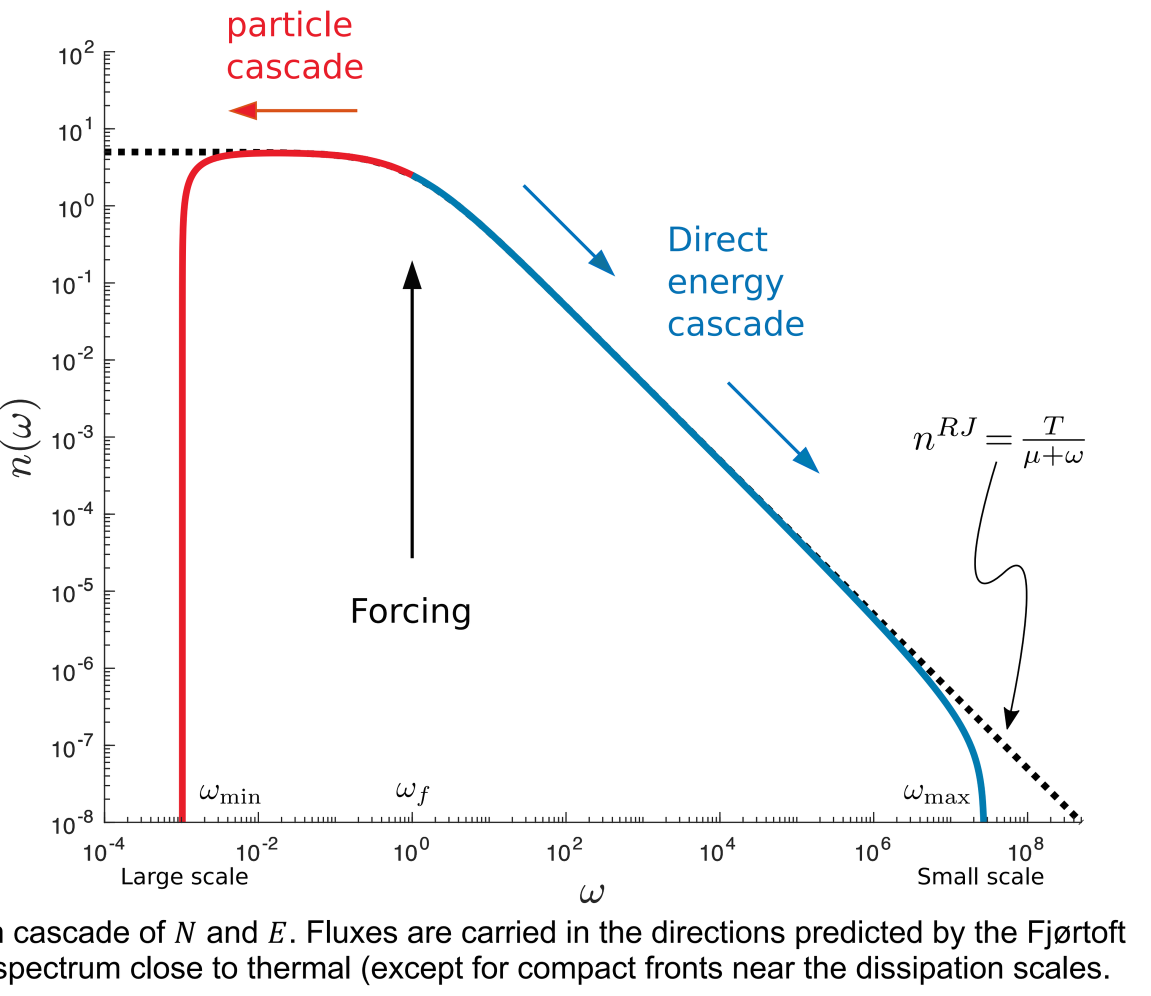

Fig 3 Dual warm cascade of N and E. Fluxes are carried in the directions predicted by the Fjørtoft argument, on a spectrum close to thermal (except for compact fronts near the dissipation scales.

Conclusions and outlook

- Warm cascades carry particles to large scale in the SNE.
- This could be a mechanism for the accumulation of dark matter at large scales in the early Universe, and relate to structure formation.
- The results from the DAM need to be made quantitative by direct numerical simulations of the SNE.
- Prospects for testing the WT of the SNE in the lab by nonlinear optics, to which eqs (1) and (2) also apply.