Singularity turbulence

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Motivation-context

- Role (and in fact existence) of singularities (or at least extreme events) in turbulence
- Extreme events (vorticity collapses) have been predicted and observed numerically (Siggia-Pumir 1987, Brachet et al 1992 for instance) and experimentally (Meneveau-Sreenivasan 1987). They are expected to be responsible of the intermittence in turbulence
- Difficult to handle in fluid flows: can we investigate a simpler model where we know the singularities (burgulence, Bec-Frisch 2000').

Focusing NLS

$$\imath \frac{\partial \psi}{\partial t} = -\frac{\alpha}{2}\nabla^2\psi - g|\psi|^{2n}\psi$$

- $\psi(\mathbf{x}, t)$ is a complex field
- the NLS equation is a model for BEC, shallow-water and nonlinear optics.
- Hamiltonian structure and mass conservation

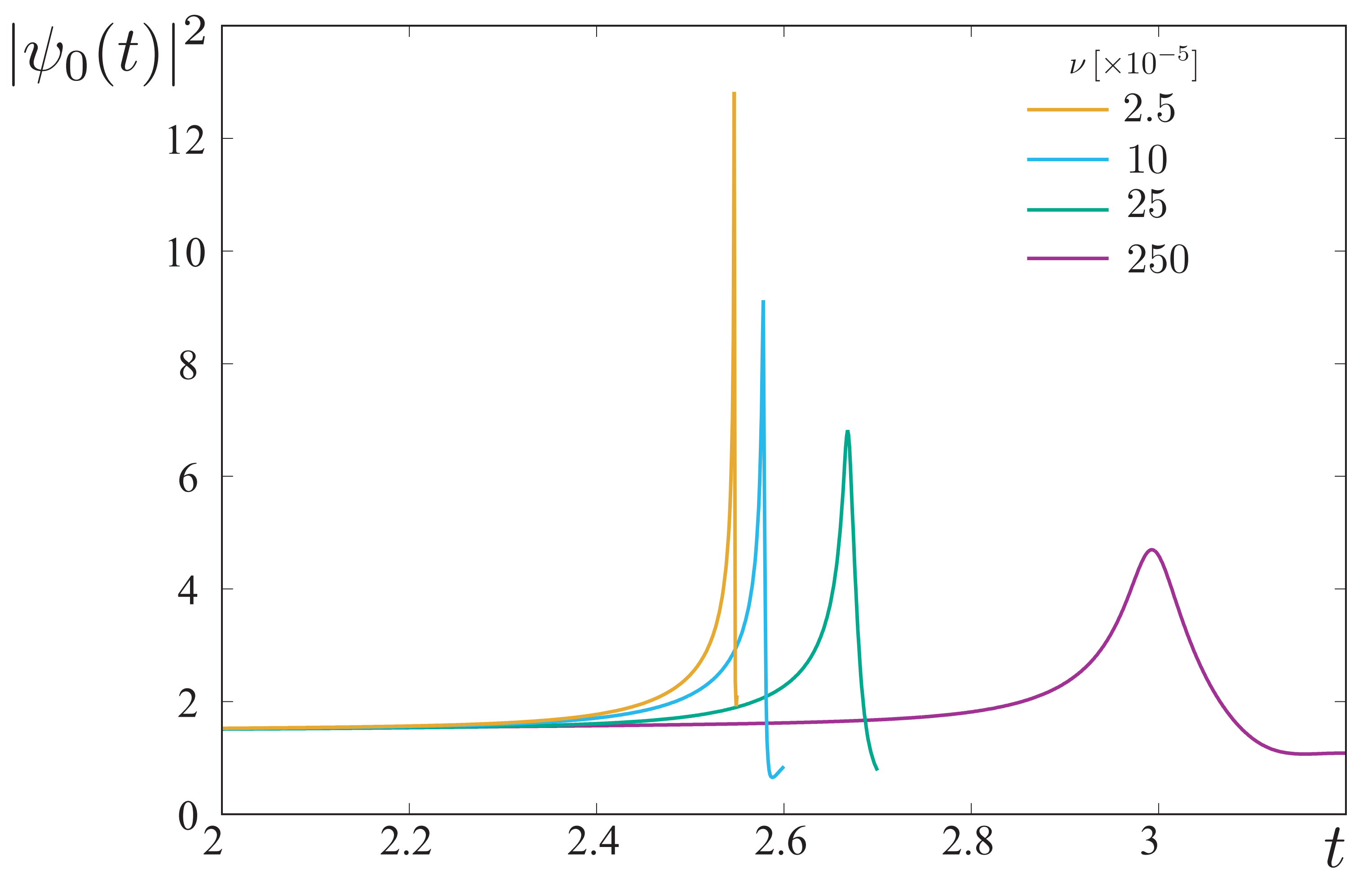
$$N = \int |\psi|^2 d\mathbf{x}$$

$$H = \int \left(\frac{\alpha}{2}|\nabla\psi|^2 - \frac{g}{n+1}|\psi|^{2(n+1)} \right) d\mathbf{x}$$

Singularity in NLS

- for the focusing NLS (g=1), the dynamics exhibits finite time singularities for nd>2 (see for instance Le Mesurier et al 1988)

- this finite time singularity is suppressed in the presence of dissipation



$$\imath\frac{\partial\psi}{\partial t} = -\frac{\alpha}{2}\nabla^2\psi - g|\psi|^{2n}\psi - \imath\nu\Delta^2\psi + f_{k_0}(\mathbf{x},t).$$

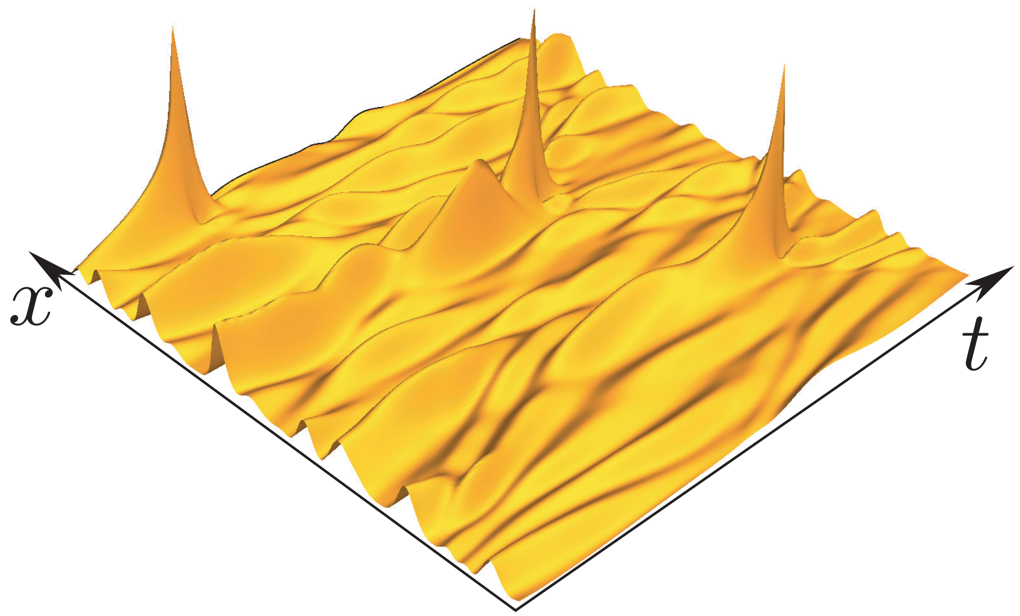
- with injection at large scale: turbulence of singularity or collapses (Dyachenko et al 1992,M. Bartuccelli et al 1990)

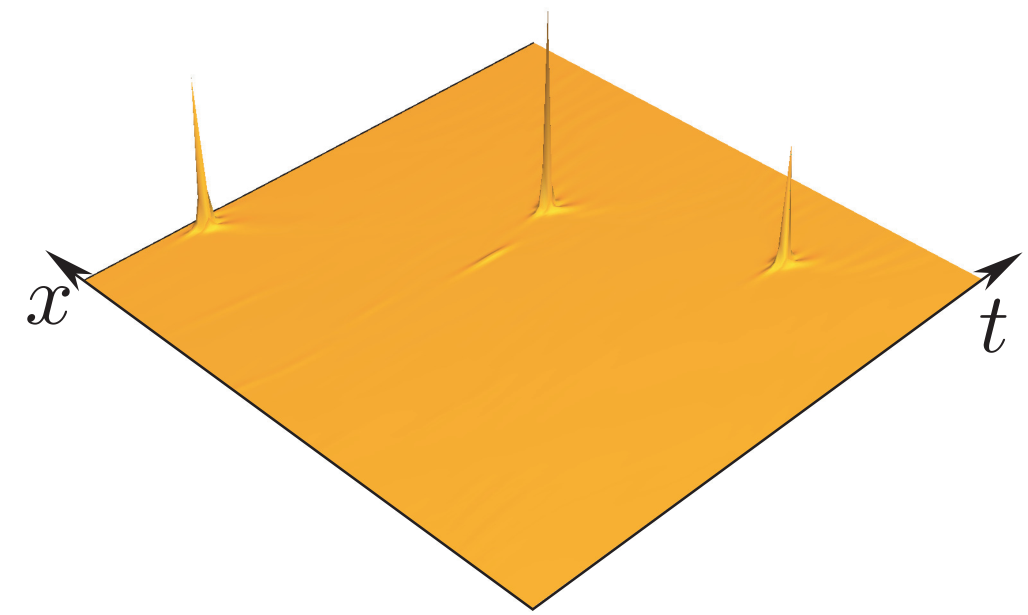
- important difference: here the mass (positive definite) is the pertinent quantity for the turbulence behavior

- wave turbulence (often observed in NLS equations) would suggest *inverse* cascade of mass and direct of energy

- focus here on the 1D case with n=3 (work in progress in 2 and 3D)

- numerical simulations



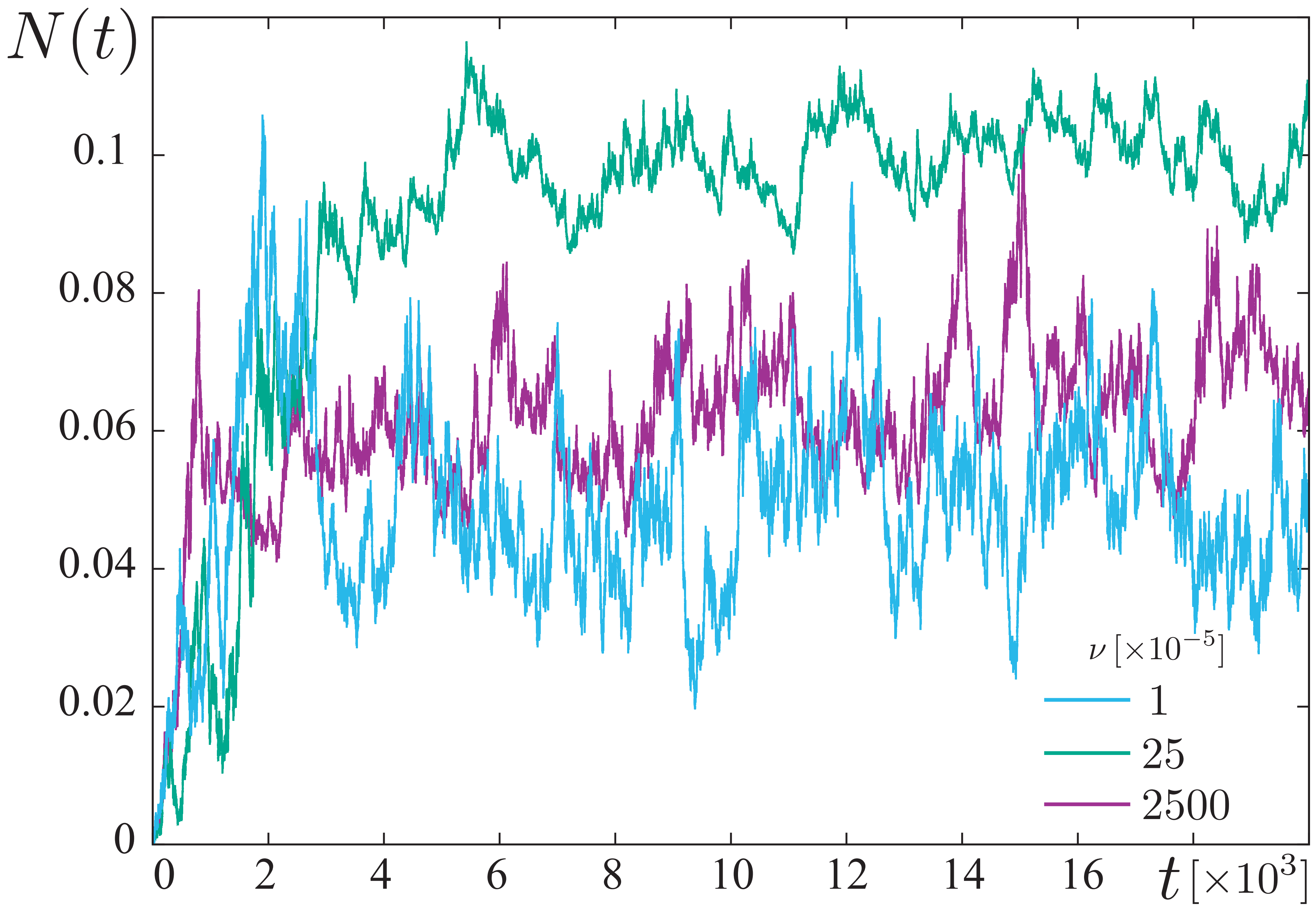


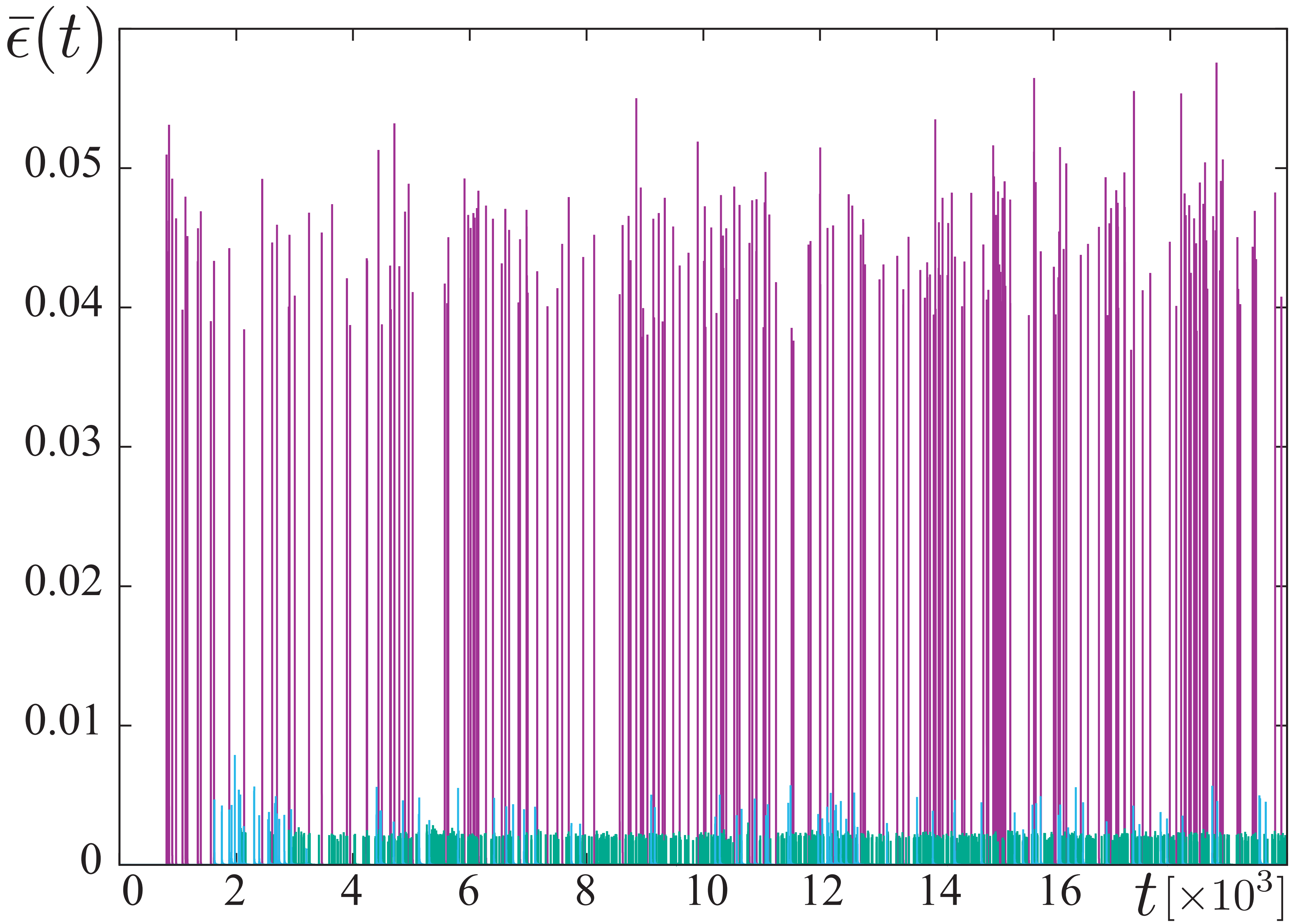
- « singularity » collapse or peak is followed by wave emission

- « dissipation » of mass is concentrated at short scale on the peaks:

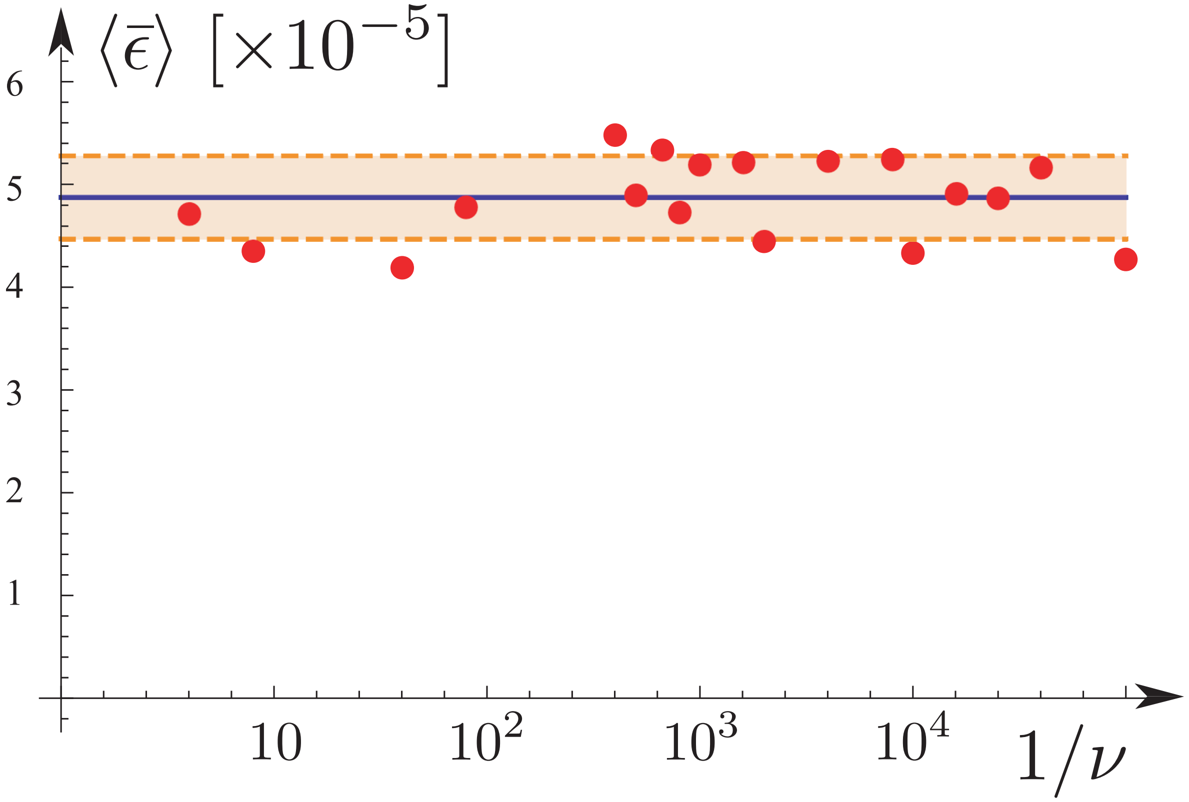
$$\frac{dN}{dt} = -2\nu \int |\Delta\psi|^2 \, d^D\mathbf{x} + i \int \left(\psi \bar{f}_{k_0} - \bar{\psi} f_{k_0}\right) \, d^D\mathbf{x}.$$

Varying only the viscosity





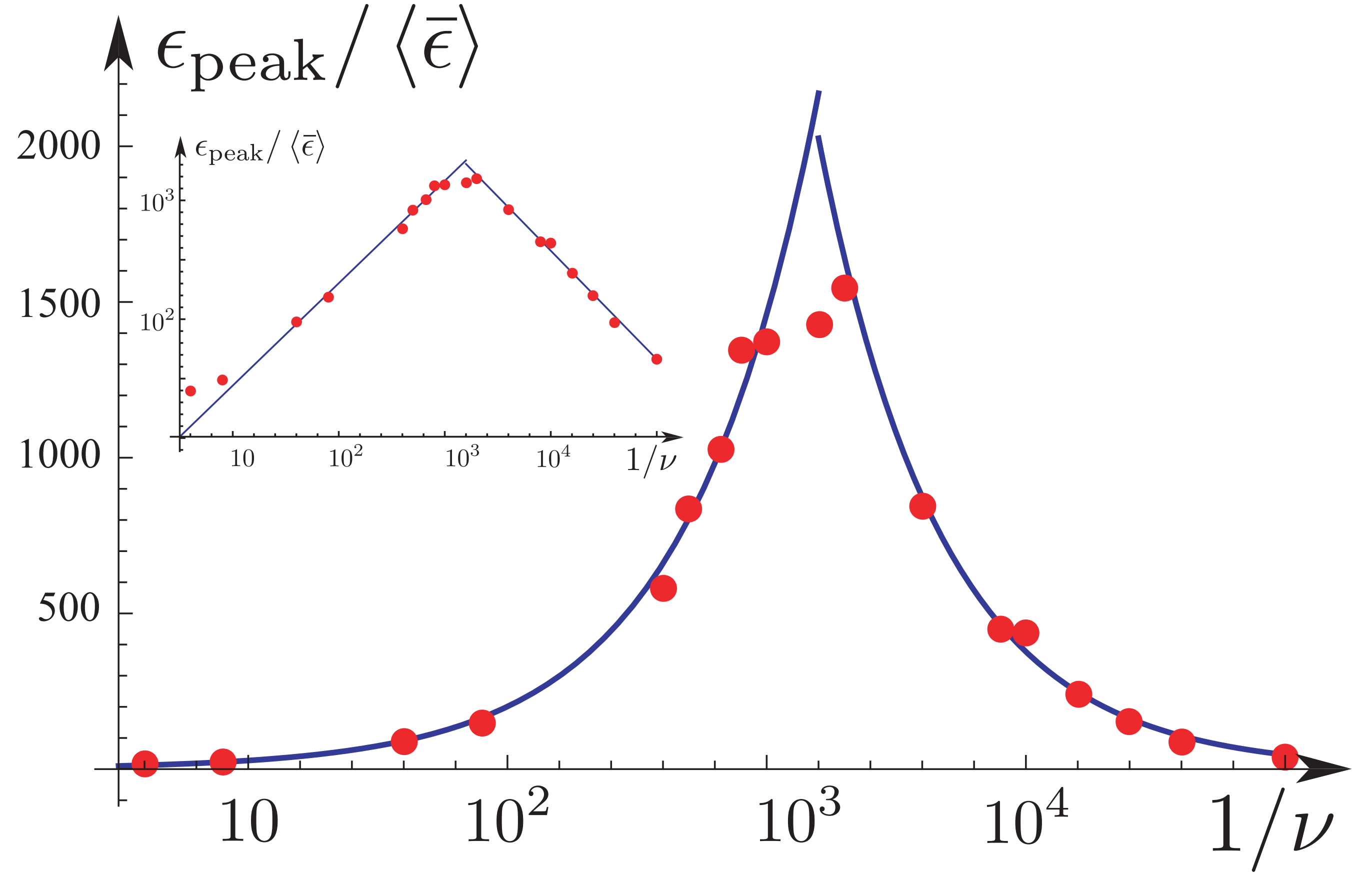
$\bar{\epsilon}(t)$
0.05
0.04
0.03
0.02
0.01
0
0
2
4
6
8
10
12
14
16
$t\,[\times 10^3]$



Anomalous dissipation?

Warning:

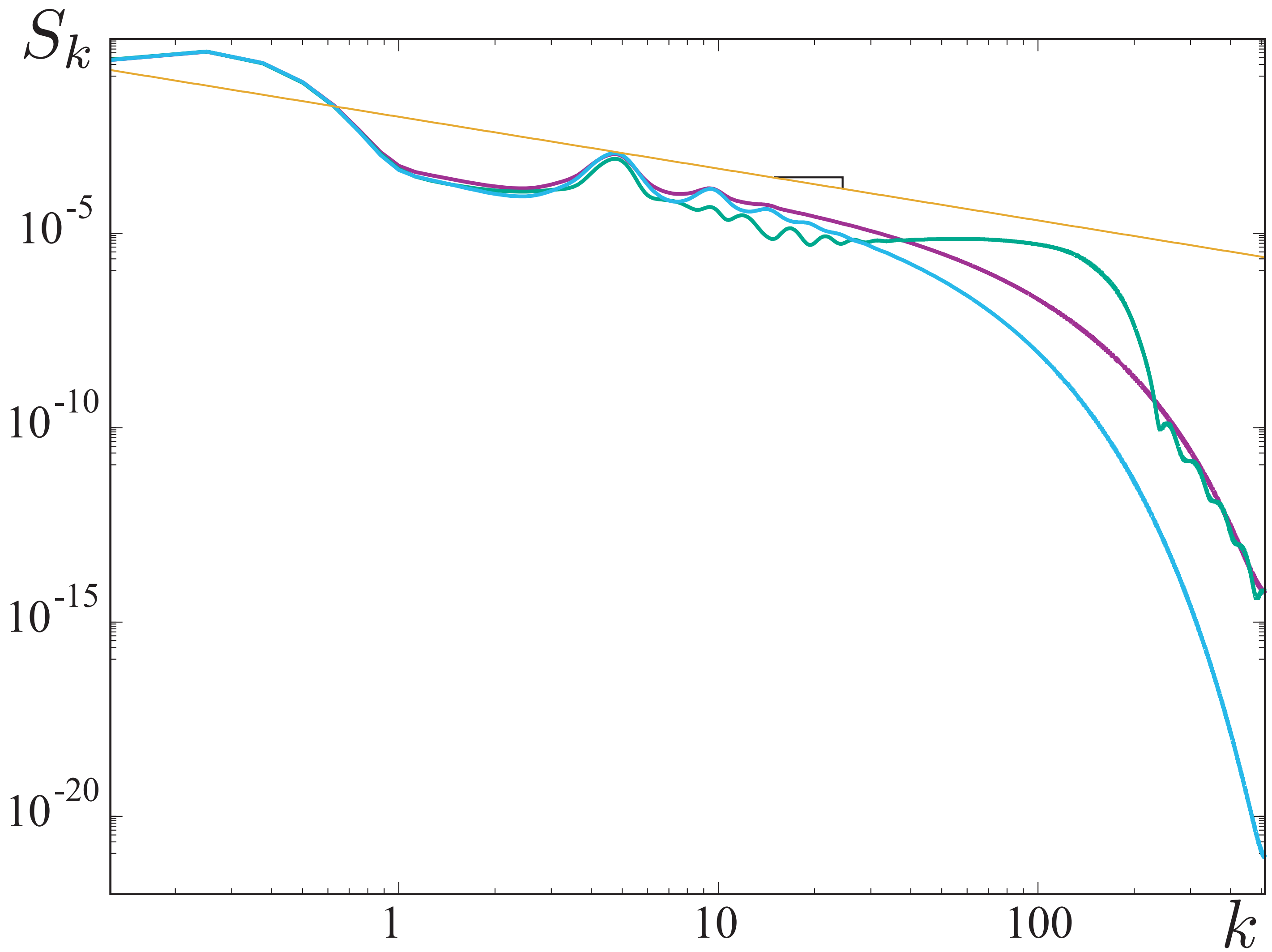
$\nu \times |\Delta\psi|^2$



Spectrum

$$S_k(t) \equiv |\hat{\psi}_k|^2 + |\hat{\psi}_{-k}|^2 \qquad \frac{1}{L}\int |\psi|^2 d\mathbf{x} = \int |\hat{\psi}_k|^2 d\mathbf{k}$$

- Spectrum fluctuates at collapse



$$\frac{1}{t} \sim \frac{1}{x^2} \sim |\psi|^6$$

$$|\psi|^2 \sim x^{-2/3}$$

$$|\psi_k|^2 \sim k^{-4/3}$$

- spectrum of the self similar collapse $\quad S_k \propto k^{-4/3}$

- Transport equation for the spectrum

$$\frac{\partial S_k}{\partial t} = -\frac{\partial Q_k}{\partial k} - 2\nu k^4 S_k + F_k$$

- Need to consider averaged in time spectra for which we have

$$\frac{\langle \partial S_k \rangle}{\partial t} = 0$$

- In the inertial range we obtain also

$$\langle Q_k \rangle = -2\nu \int_k^{\infty} k^4 \langle S_k \rangle dk \equiv \langle \epsilon \rangle$$

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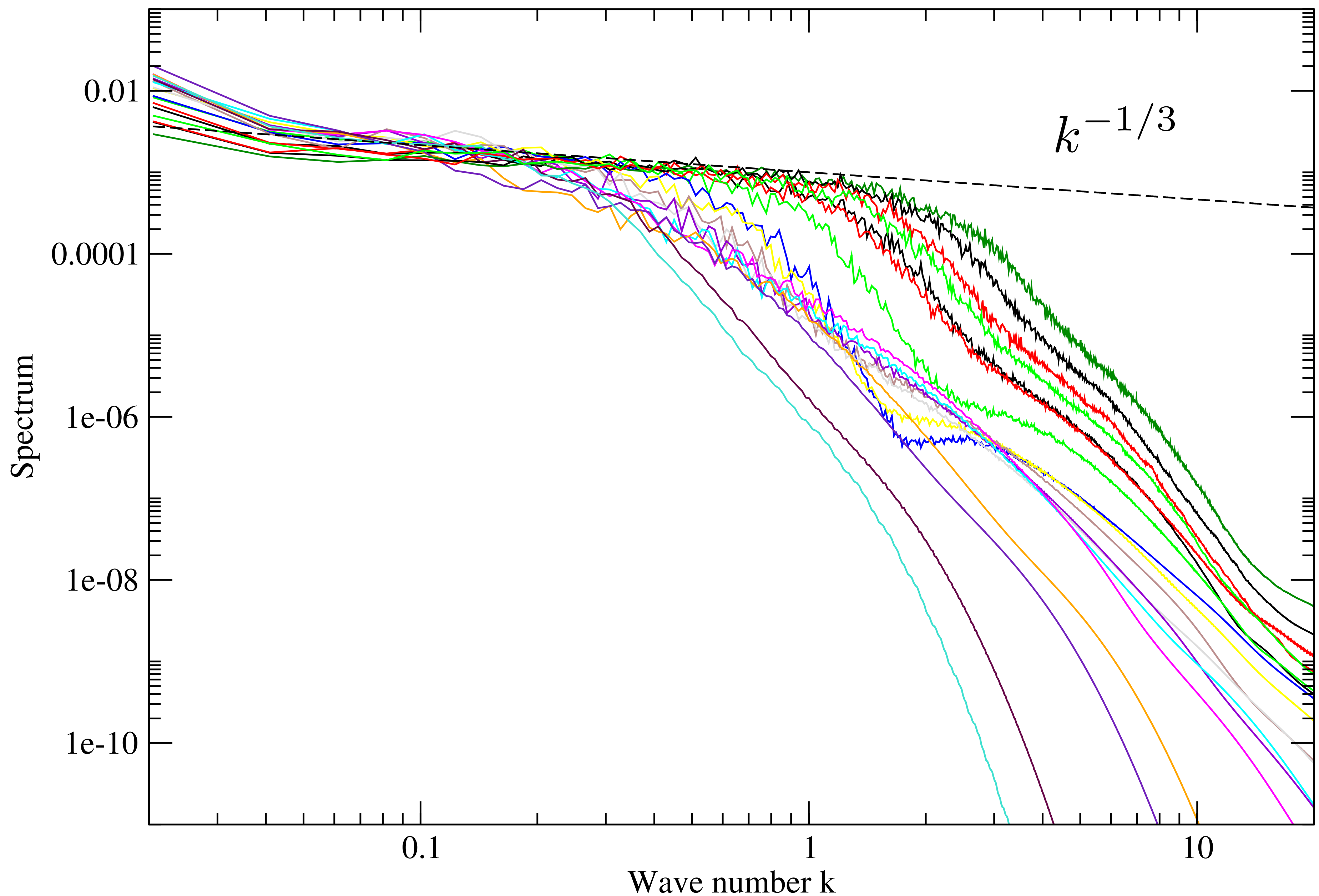
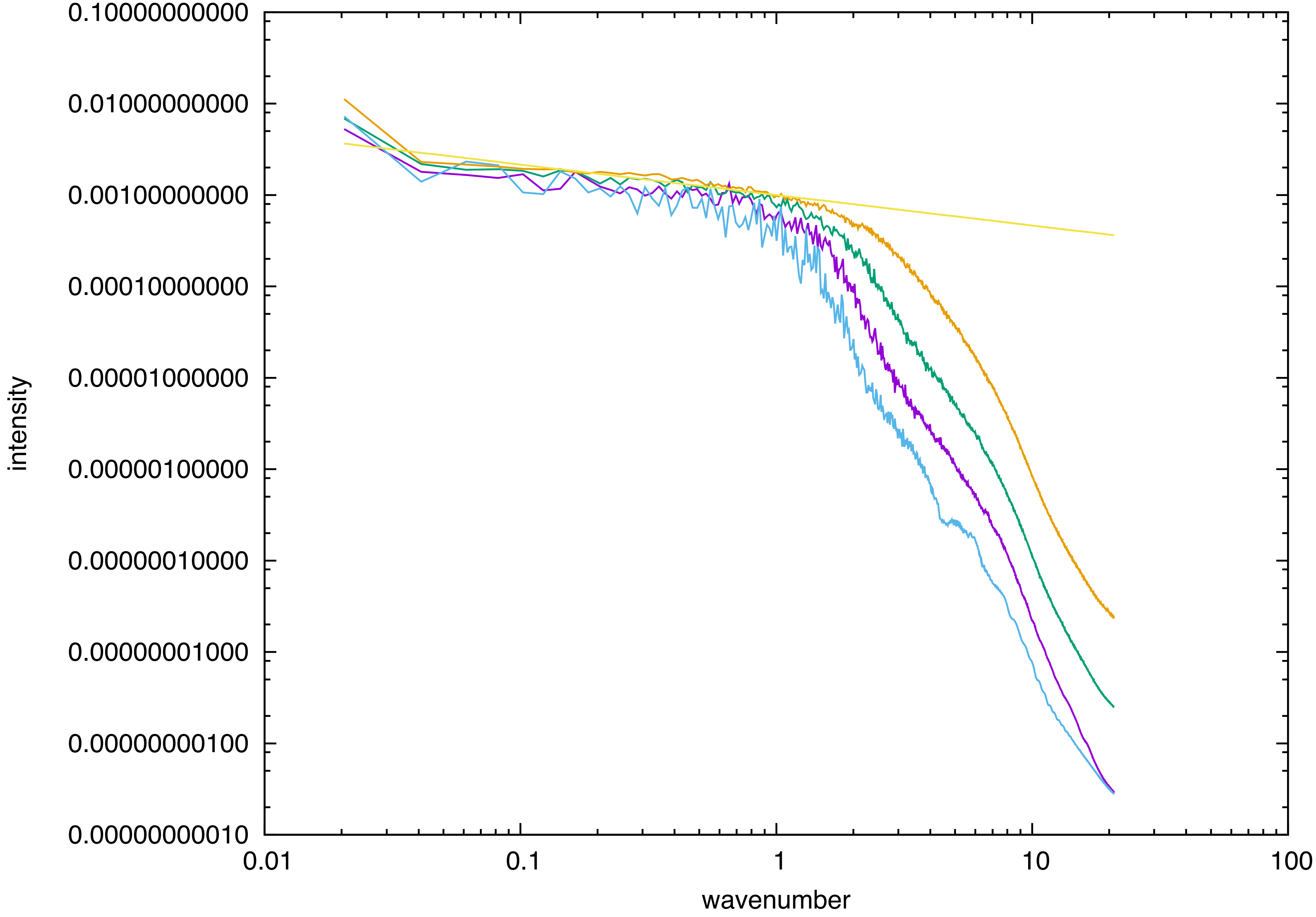
Spectrum expands in k as the viscosity decreases. Its amplitude seems independent of the injection rate



Varying the injection



Phillips spectrum?

Kolmogorov-like scaling analysis

$$[S_k] = \rho\ell \qquad [\epsilon] = \rho\tau^{-1} \qquad [\alpha] = \ell^2\tau^{-1} \qquad [g] = \rho^{-3}\tau^{-1}$$

$$\langle S_k \rangle = \frac{\langle \bar{\epsilon} \rangle}{\alpha k^3} F\left(\frac{\alpha k^2}{\left(g \langle \epsilon \rangle^3\right)^{\frac{1}{4}}} \right)$$

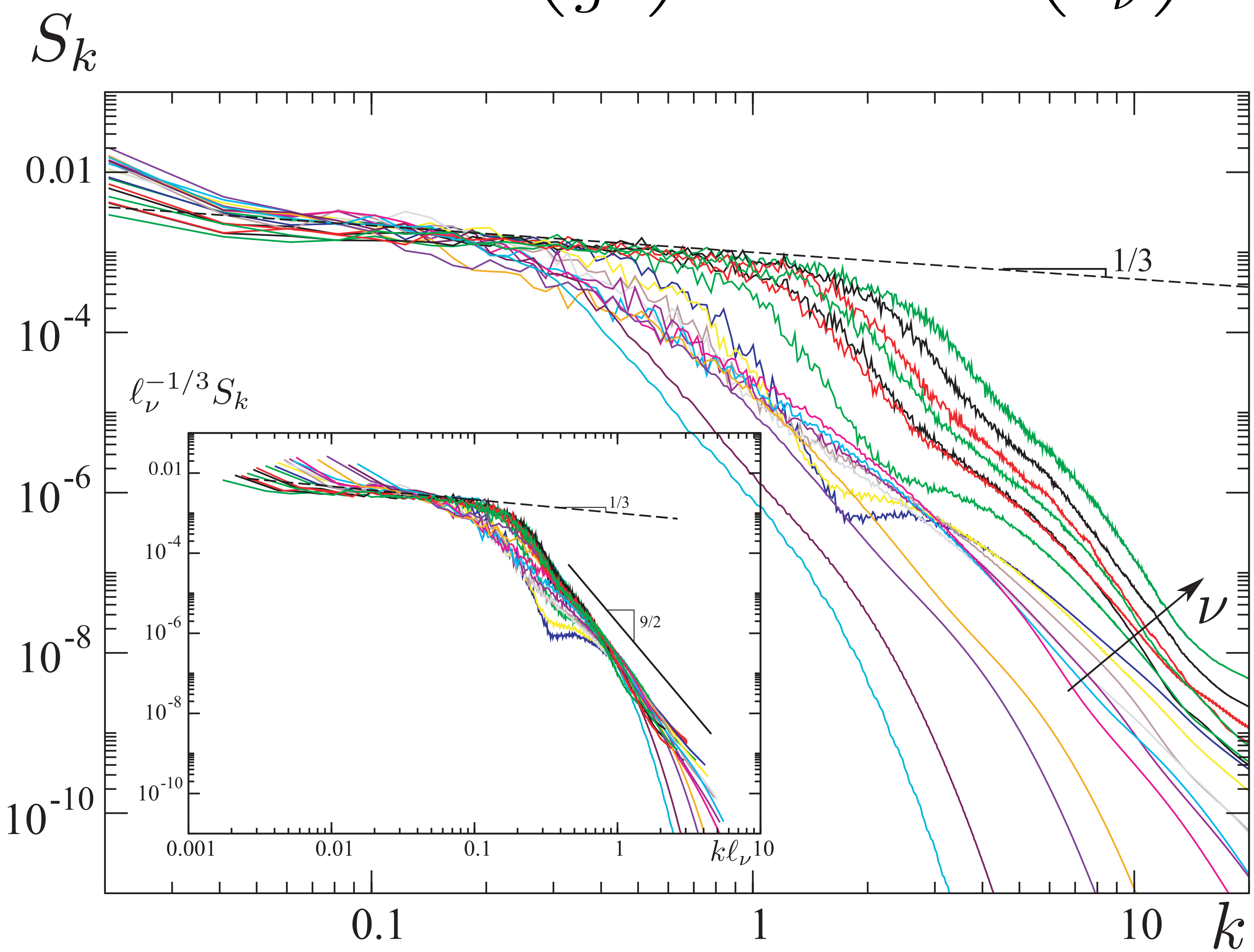
If we look for a solution independent of the injection rate

$$\langle S_k \rangle \propto \left(\frac{\alpha}{g^4} \right)^{1/3} k^{-1/3}$$

Kolmogorov scale $\ell_\nu \sim \left(\frac{\alpha \nu^3}{g \bar{\epsilon}^3} \right)^{1/14}$ $\qquad k_\nu \sim \left(\frac{g \bar{\epsilon}^3}{\alpha \nu^3} \right)^{1/14}$

Suggest the following self-similar scaling for the spectrum

$$\langle S_k \rangle = \left(\frac{\alpha}{g^4}\right)^{1/3} k_\nu^{-1/3} G\left(\frac{k}{k_\nu}\right)$$



Intermittency-structure functions

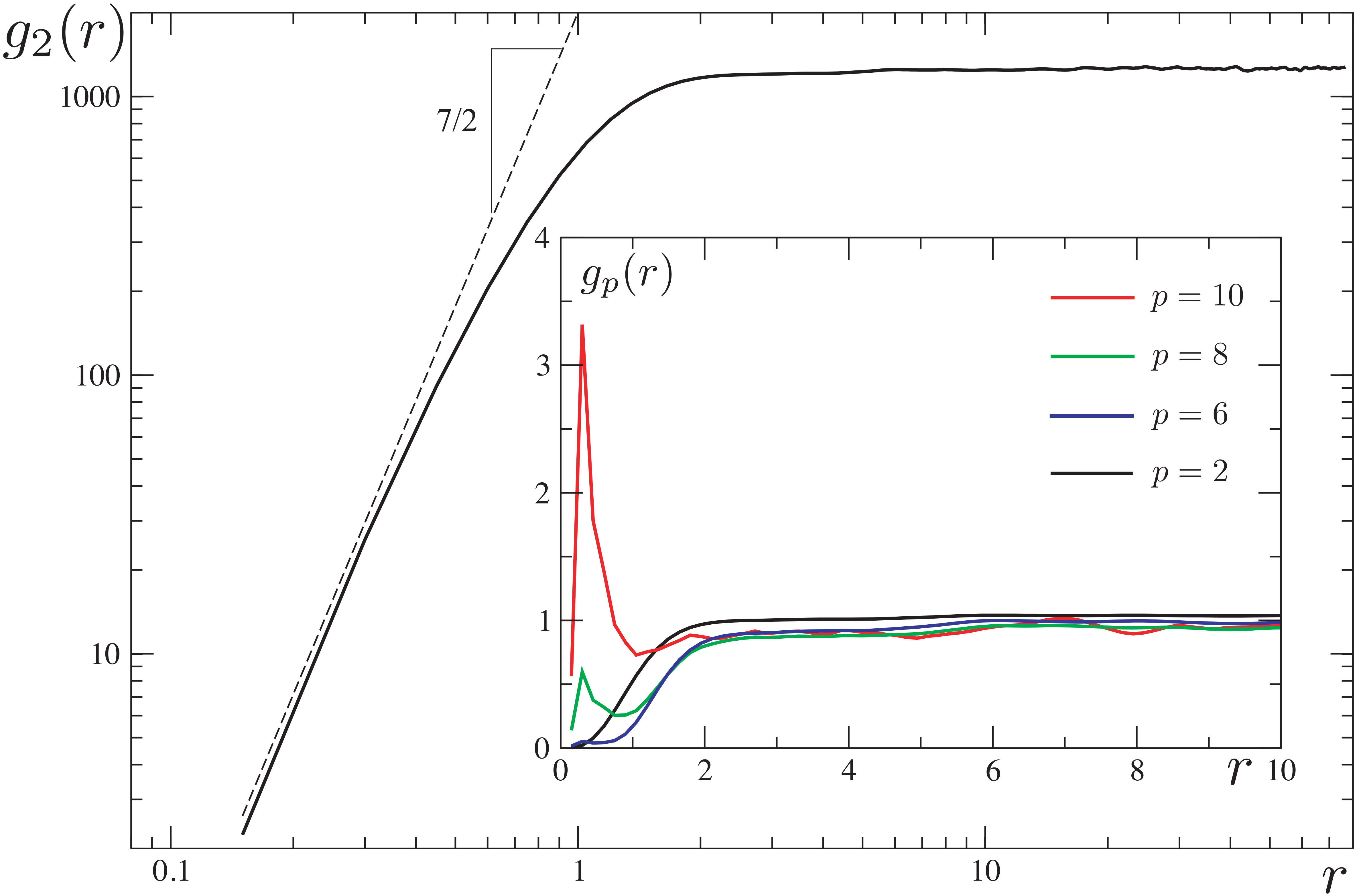
$$g_p(r) = \overline{|\psi(x+r) + \psi(x-r) - 2\psi(x)|^p}$$

p=2 can be deduced from the spectrum scalings

$$g_2(r) \sim r^{7/2}$$ at short scales

$$g_2(r) \sim r^{-2/3}$$ inertial range

High p's should witness the singularity at small scales


$g_2(r)$
1000
100
10
0.1
1
10
r
7/2
$g_p(r)$
$p = 10$
$p = 8$
$p = 6$
$p = 2$
0
1
2
3
4
6
8
10

Conclusion

- « singularity » mediated turbulence (in the spirit of « defect » mediated (Coullet, Gil & Lega 1989) is observed (singularity cured by viscosity) in a version of the focusing NLS
- simple model where singularity in the inviscid limit is known. Mass « cascade »
- dissipation of mass concentrated in the collapses
- Kolmogorov like spectra are observed (needs additional condition for the exponent, different that those of the collapse and of the WTT, possibility of Phillips spectrum)
- singularity manifests through intermittency at high order

Thanks! Questions?