Norwich – October 30, 2019

Quantum vortex dynamics by geometric and topological methods

RENZO L. RICCA

Department of Mathematics & Applications, U. Milano-Bicocca

renzo.ricca@unimib.it

Joint work with S. Zuccher (U. Verona) & M. Foresti (UniMiB)

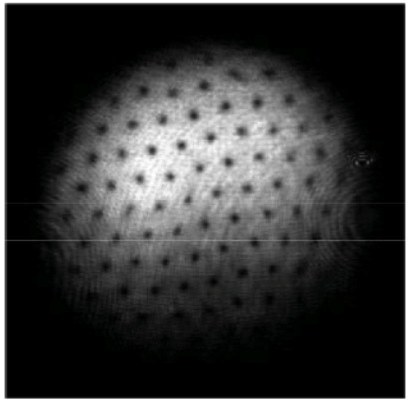
- *Gross-Pitaevskii (1961-1963) equation:* $\psi = \psi(\mathbf{x}, t)$

$$\frac{\partial \psi}{\partial t} = \frac{i}{2}\nabla^2\psi + \frac{i}{2}(1 - |\psi|^2)\psi, \quad \text{with } |\psi|^2 \to 1 \text{ as } |\mathbf{x}| \to \infty .$$

- *Madelung (1926) transformation:*

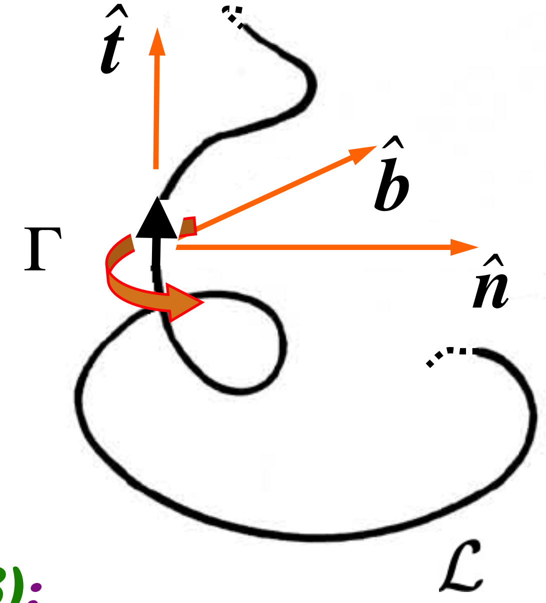
$$\psi = \sqrt{\rho}\exp(i\chi) \quad \begin{cases} \rho = |\psi|^2 \\ \mathbf{u} = \nabla\chi \end{cases}$$

- *Hydrodynamic interpretation of GPE*



(Anderson et al. 1995)

Helicity and linking numbers



- ***Helicity*** H **:**

$$H = \int_{V(\boldsymbol{\omega})} \mathbf{u}\cdot\boldsymbol{\omega}\, dV \,,$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ ***with*** $\nabla\cdot\mathbf{u} = 0$ ***in*** $\mathbb{R}^3$.

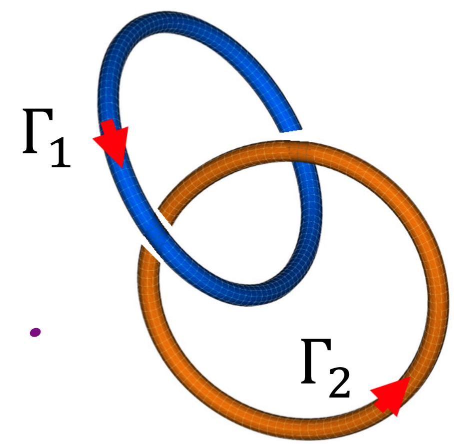
- ***Under GPE (Salman 2017; Kedia et al. 2018):***

$$H_{GPE} = \Gamma \oint_{\mathcal{L}} \mathbf{u}\cdot d\mathbf{l} = \Gamma \oint_{\mathcal{L}} \nabla\chi\cdot d\mathbf{X} = 0 \,.$$

- ***Theorem (Moffatt 1969; Moffatt & Ricca 1992). Let*** $\mathcal{L}_n$ ***be a disjoint union of*** n ***vortex tubes in an ideal fluid.***

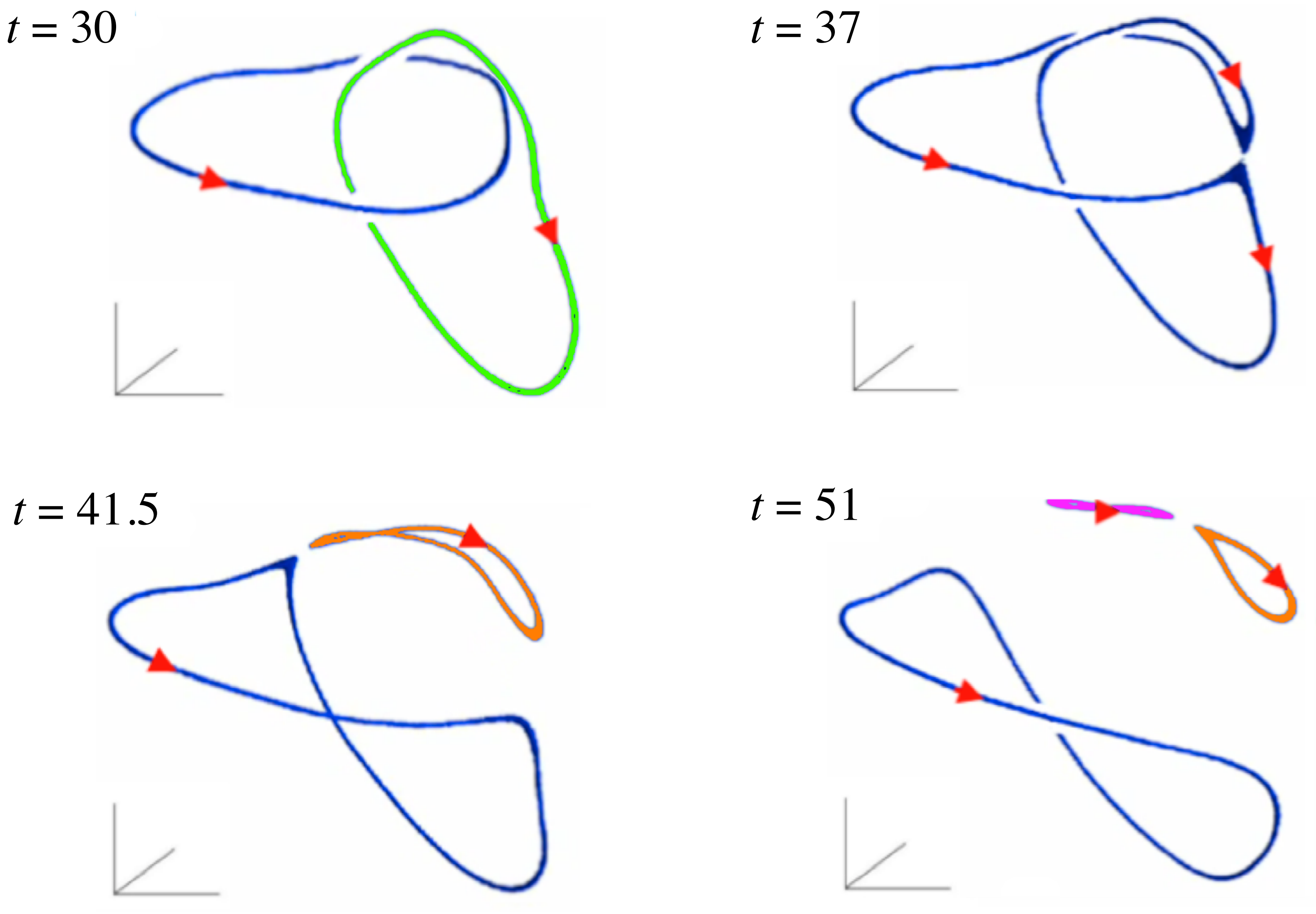
$$H_{GPE} = \underbrace{\cancel{\int_{V(\boldsymbol{\omega})} \mathbf{u}\cdot\boldsymbol{\omega}\, dV}}_{=0} = \sum_{i\neq j} Lk_{ij}\Gamma_i\Gamma_j + \sum_i Lk_i\Gamma_i^2$$

$$= \sum_{i\neq j} Lk_{ij}\Gamma_i\Gamma_j + \sum_i \left(Wr + Tw\right)_i \Gamma_i^2 \,.$$

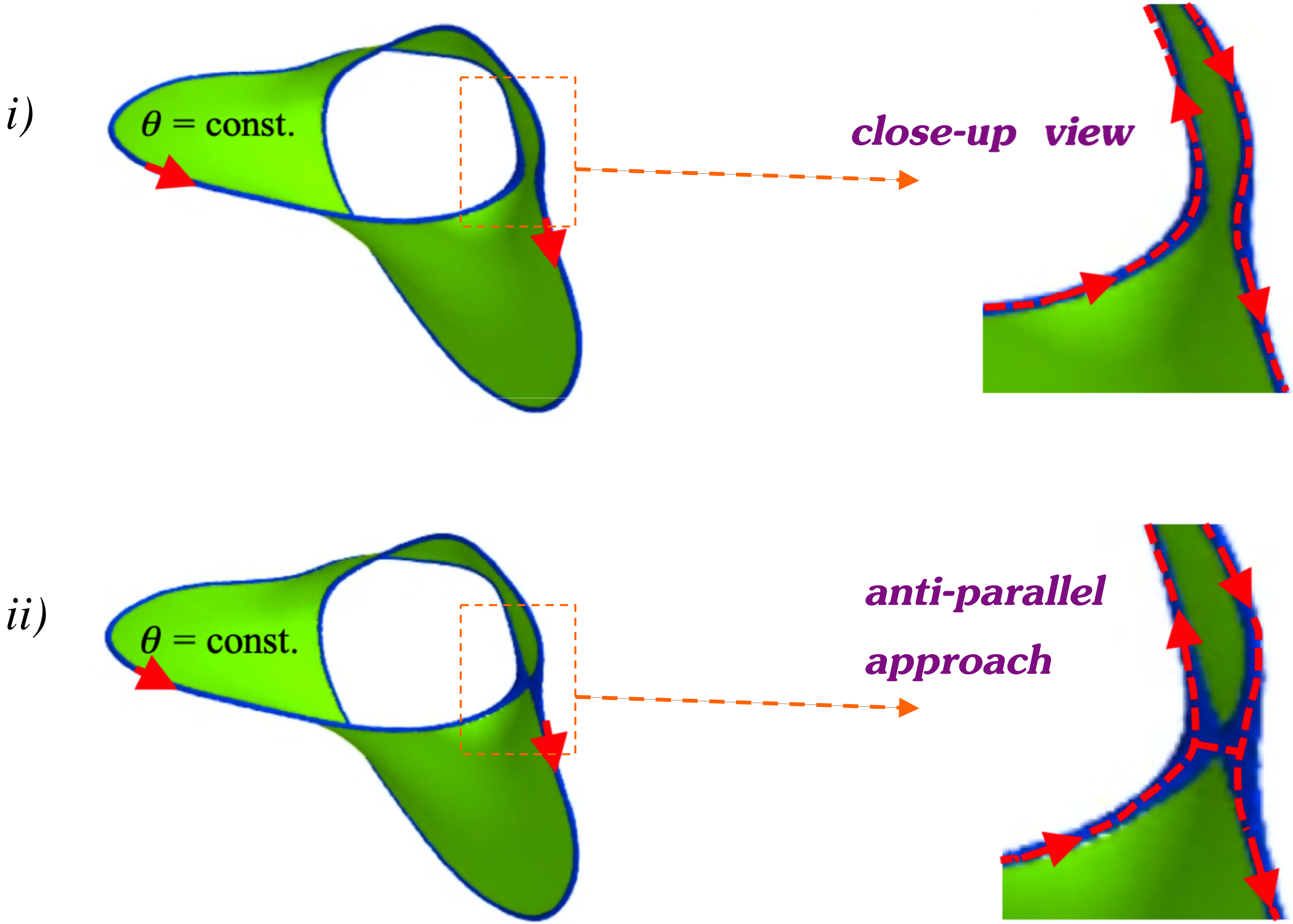


(Salman 2017)

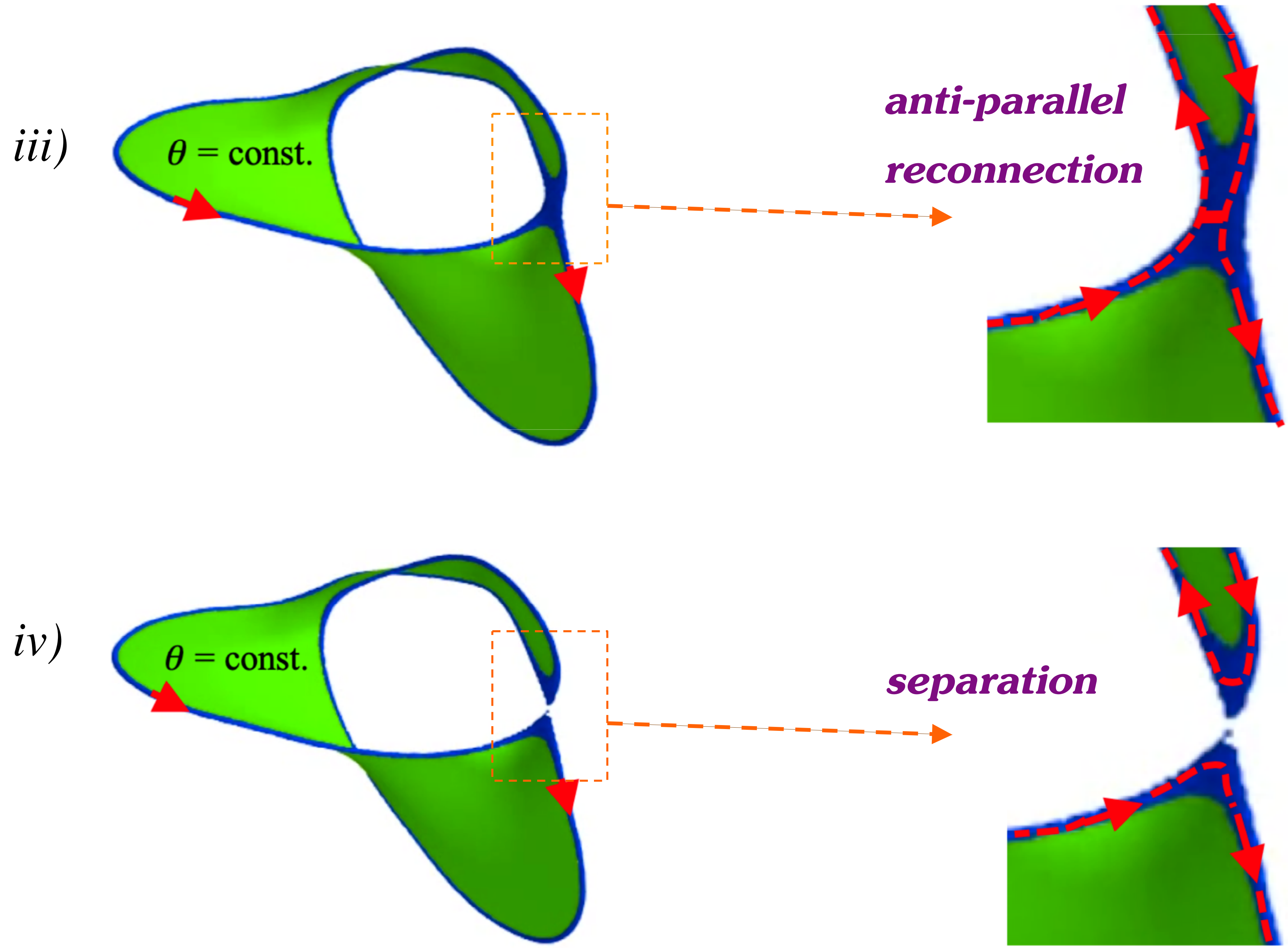
Cascade process of Hopf link ($\Gamma = 1$)



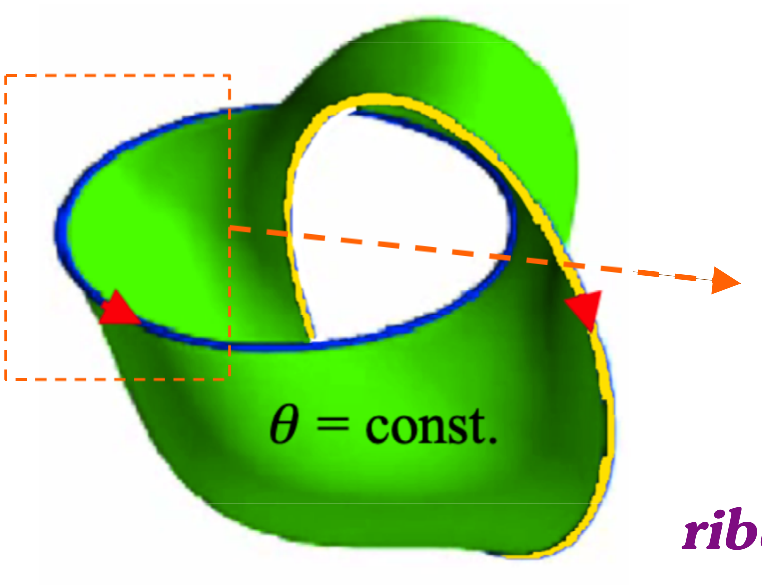
Reconnection process of iso-phase surface

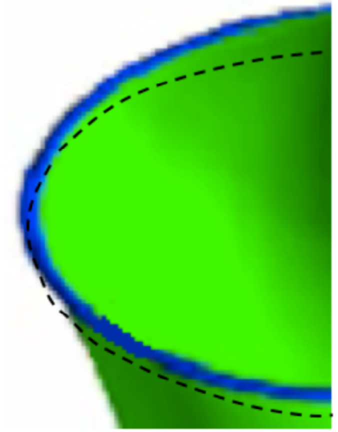


Reconnection process of iso-phase surface

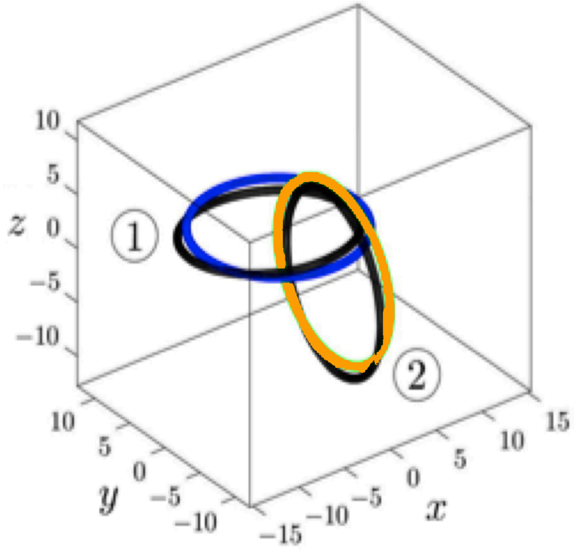


Twist analysis by isophase ribbon construction

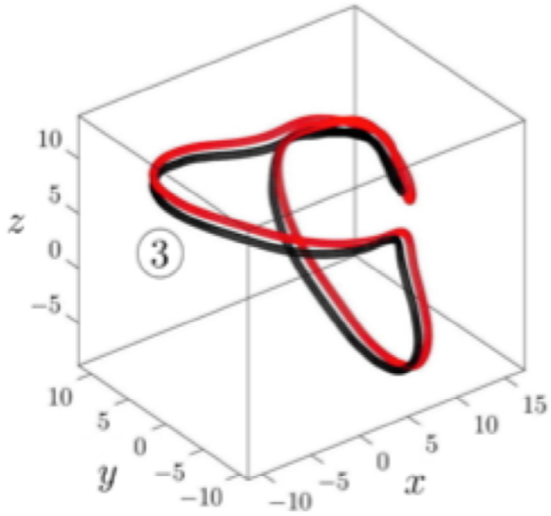




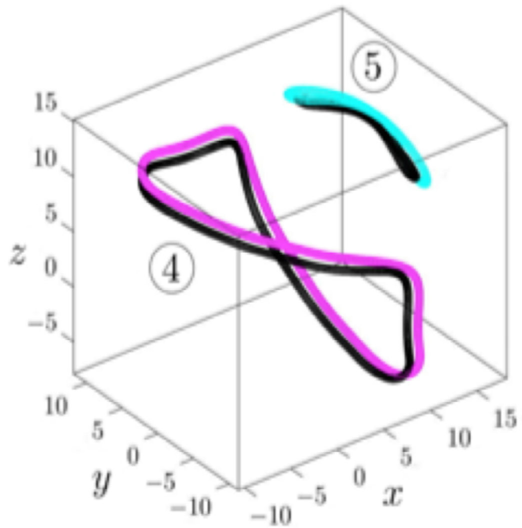
ribbon construction



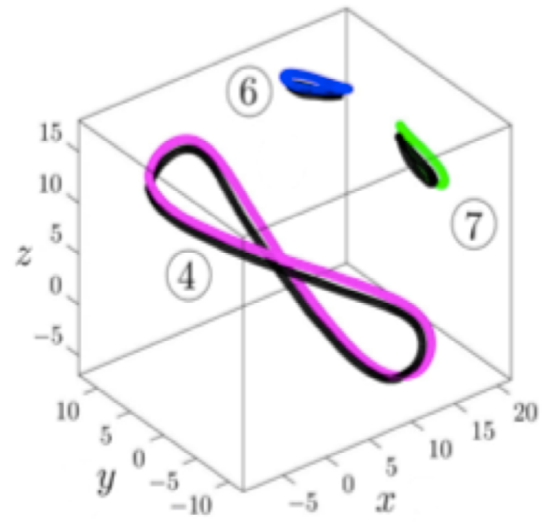
$t = 0$



$t = 38$

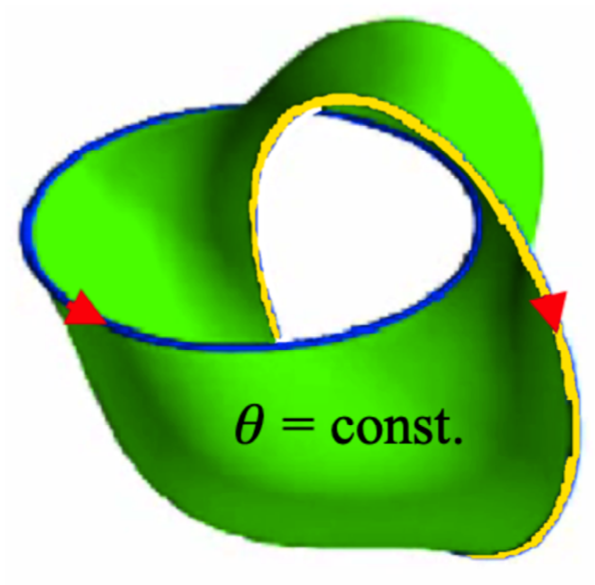


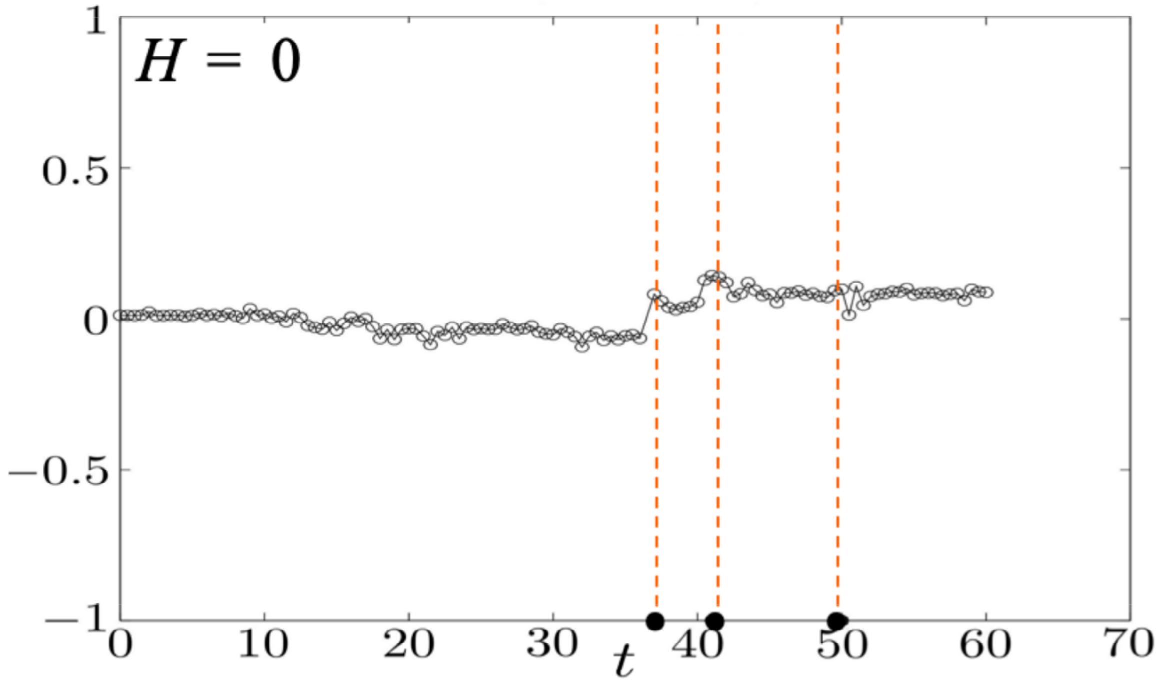
$t = 47$



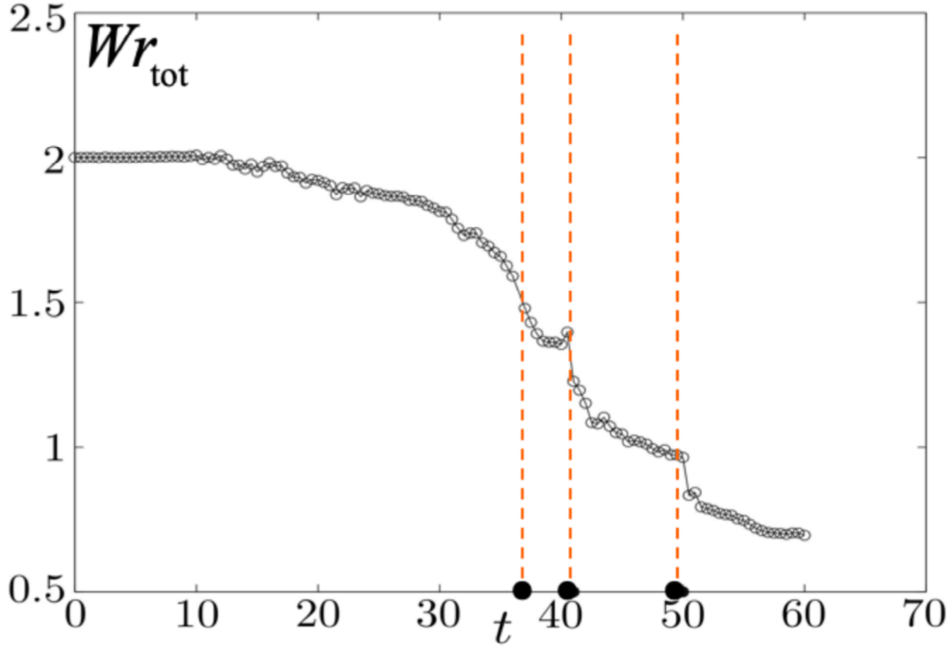
$t = 57$

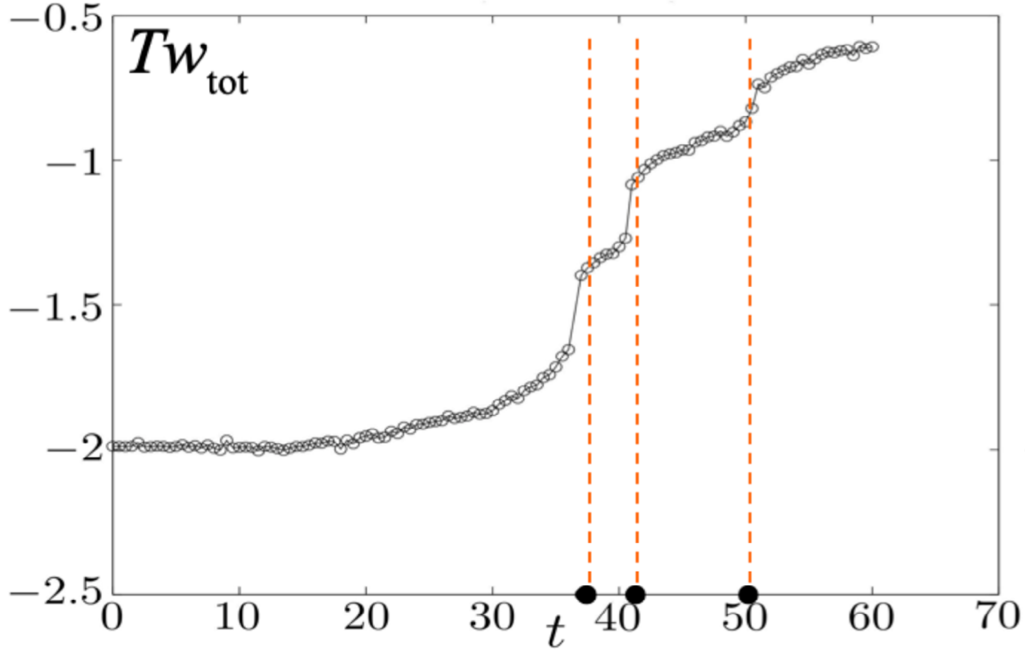
Writhe and twist contributions (Zuccher & Ricca PRE 2017)



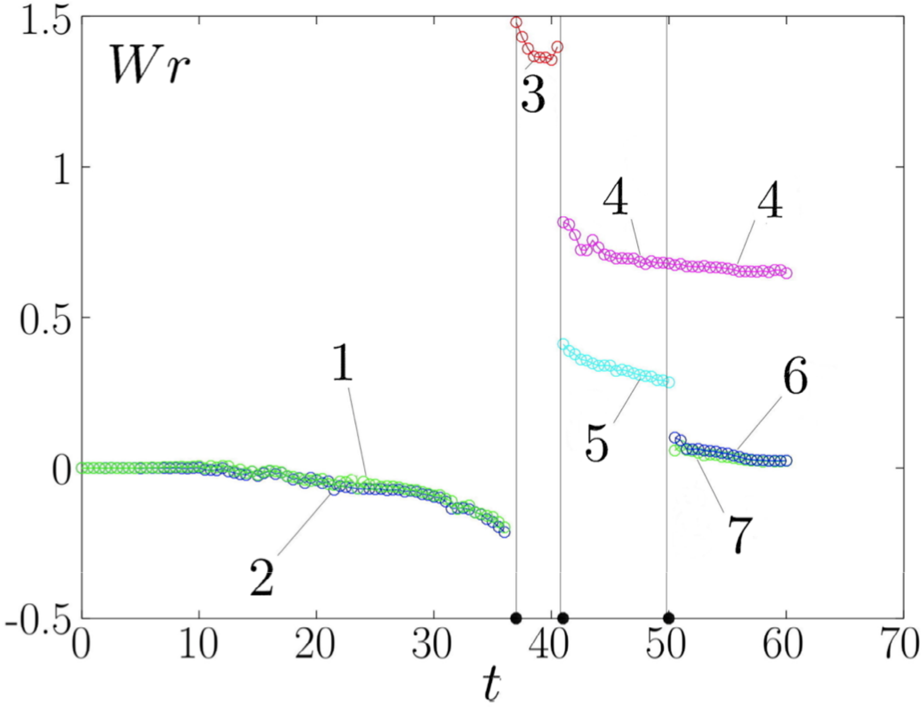


$$Wr_{\text{tot}} = Wr_1 + Wr_2 + 2Lk_{12}$$





Individual writhe and twist contributions



- ***Writhe remains conserved across anti-parallel reconnection:***

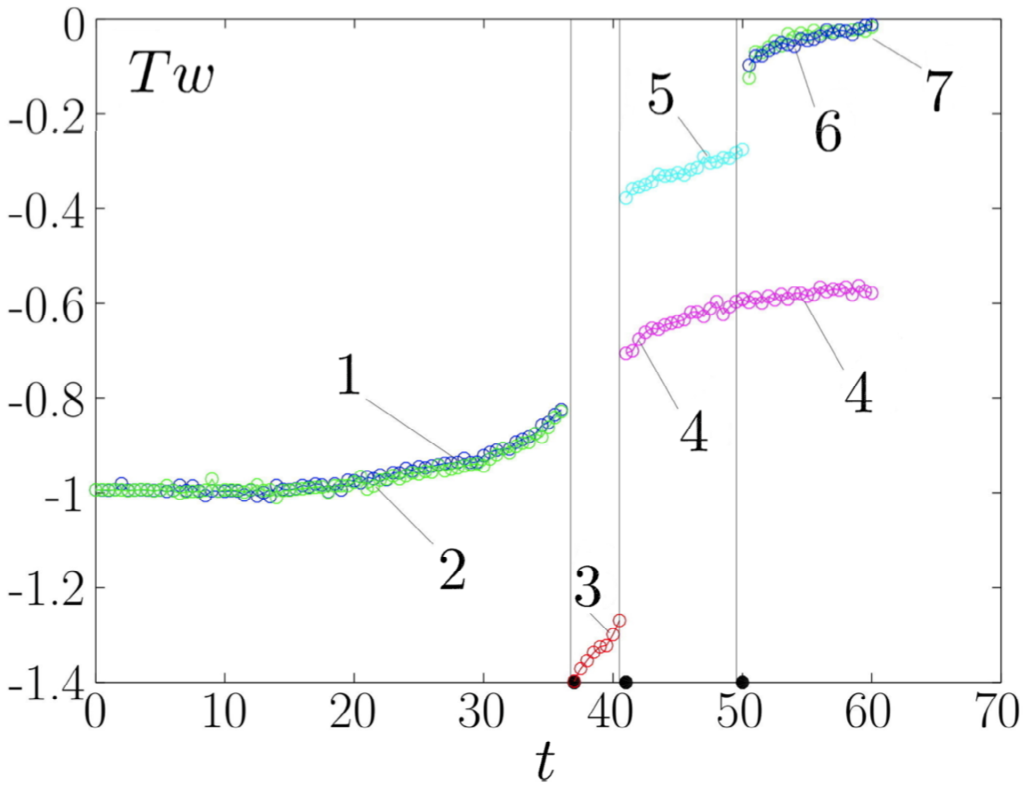
$$Wr(\mathcal{L}_1 \cup \mathcal{L}_2) = Wr(\mathcal{L}_1 \# \mathcal{L}_2)\,.$$

(Laing et al. 2015)

- ***Twist remains conserved across anti-parallel reconnection:***

$$Tw(\mathcal{L}_1 \cup \mathcal{L}_2) = Tw(\mathcal{L}_1 \# \mathcal{L}_2)\,.$$

- ***Total writhe and twist decrease monotonically during the process.***



Interpretation of momentum in terms of weighted area

Consider the linear momentum (per unit density):

$$\mathbf{P} = \frac{1}{2}\int_{V(\boldsymbol{\omega})} \mathbf{X}\times\boldsymbol{\omega}\, dV = \frac{\Gamma}{2}\oint_{\mathcal{L}} \mathbf{X}\times d\mathbf{X} = \Gamma\int_{\mathcal{A}(\mathcal{L})} \hat{\mathbf{e}}_P\, dS = cst.$$

where $\mathcal{A}(\mathcal{L})$ *is the area projected along* $\hat{\mathbf{e}}_P$ *bounded by* $\mathcal{L}$ *.*

Consider the P_i *component of* $\mathbf{P}$ *along the* i*-direction (* $i = x, y, z$ *), and* $\mathcal{A}_i = \mathcal{A}(\vec{\Lambda}_i)$ *the area of the projected graph* $\vec{\Lambda}_i$ *along* i*.*

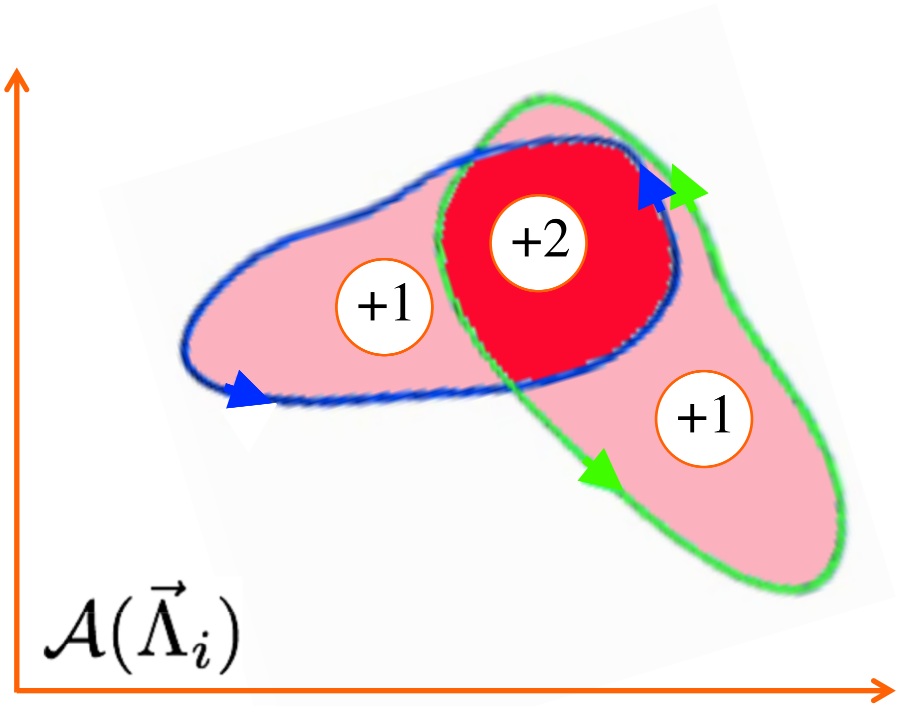
The weighted area $\mathcal{A}_i$ *is given by*

$$\mathcal{A}_i = \sum \mathcal{I}_j A_{ji}$$

where

$$\mathcal{I}_j = \mathcal{I}(R_j) = \sum_{r\in\{\hat{\boldsymbol{\rho}}\,\cap\,\vec{\partial}R_j\}} \epsilon_r$$

and $A_{ji} = A_{ji}(R_j)$ *denotes the standard area of* R_j *.*

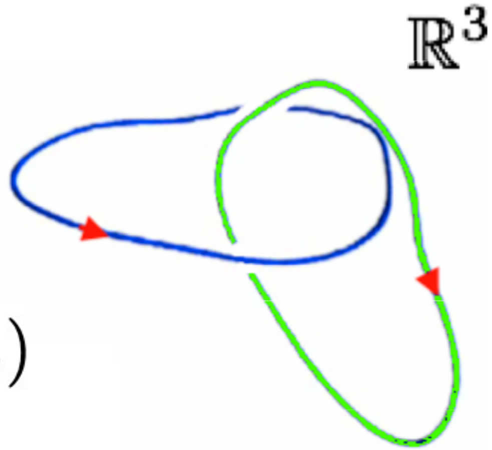


Linear and angular momentum by weighted area information

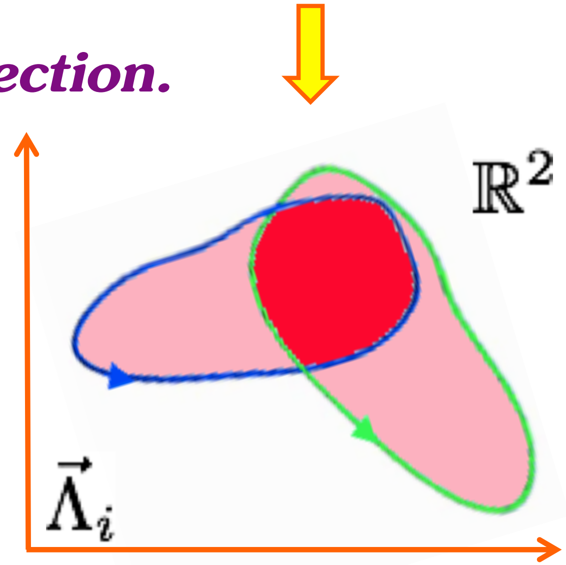
- ***Theorem* *(Ricca, 2008; 2012)*. *The linear and angular momentum* $\mathbf{P}$ *and* $\mathbf{M}$ *of a vortex link of circulation* Γ *can be expressed in terms of weighted areas of the projected graph regions by***

$$\mathbf{P} = \frac{1}{2}\int_{V(\boldsymbol{\omega})} \mathbf{X} \times \boldsymbol{\omega}\, dV = \Gamma \vec{\mathcal{A}}\,,$$

$$\mathbf{M} = \frac{1}{3}\int_{V(\boldsymbol{\omega})} \mathbf{X} \times (\mathbf{X} \times \boldsymbol{\omega})\, dV = \frac{2}{3}\Gamma \boldsymbol{\zeta} \cdot \vec{\mathcal{A}}\,,$$

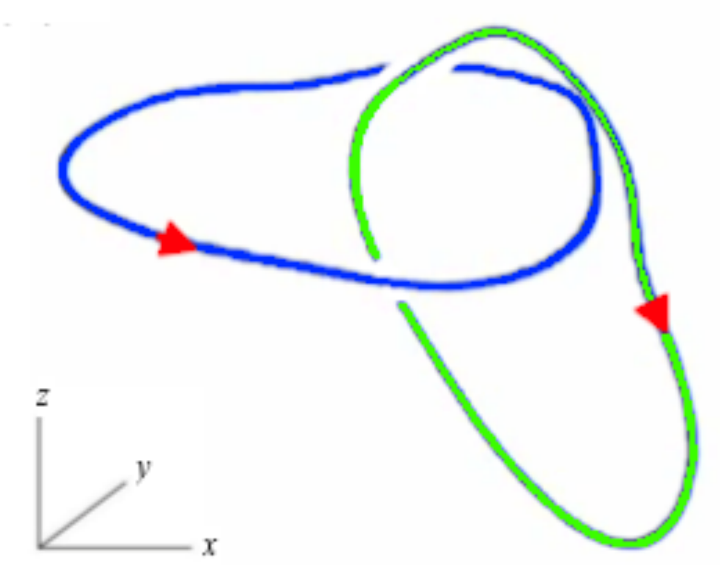


where $\vec{\mathcal{A}} = (\mathcal{A}_x, \mathcal{A}_y, \mathcal{A}_z)$, $\boldsymbol{\zeta} \cdot \vec{\mathcal{A}} = (\zeta_x \mathcal{A}_x, \zeta_y \mathcal{A}_y, \zeta_z \mathcal{A}_z)$ ***and*** $\mathcal{A}_i = \mathcal{A}(\vec{\Lambda}_i)$ ***(*** $i = x, y, z$ ***) denotes the weighted area of the projected graph*** $\vec{\Lambda}_i$ ***along the*** i***-direction.***



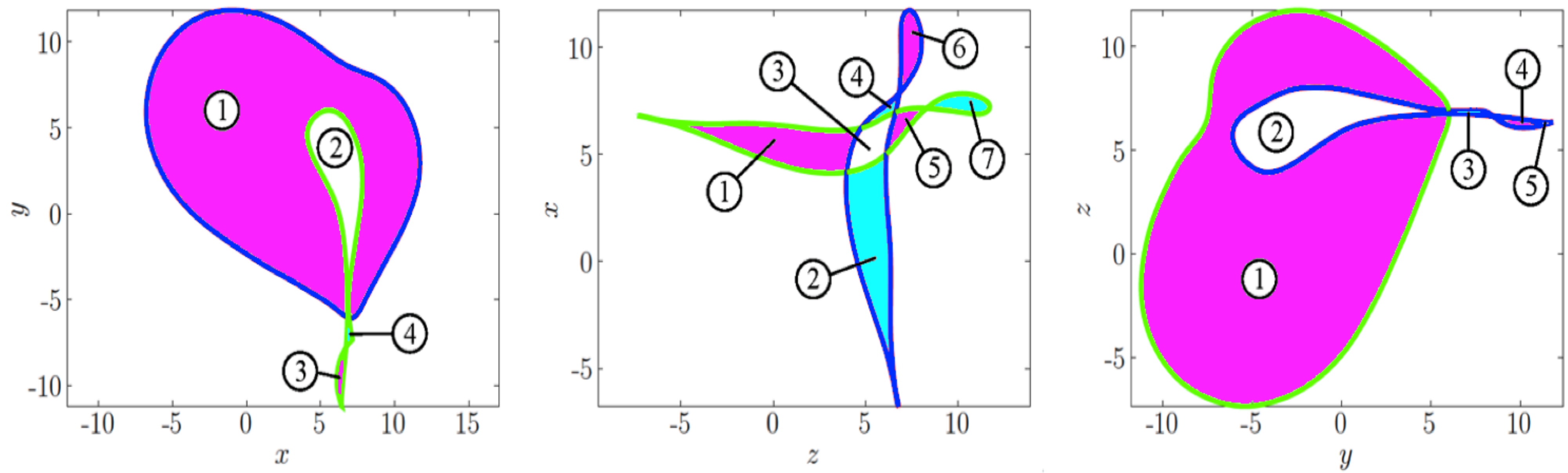
- ***Corollary.* *The components of linear and angular momentum of a vortex tangle can be computed in terms of weighted areas of the projected graph regions of the tangle.***

Weighted area computation: $t = 35$ *(Zucchher & Ricca PRE 2019)*

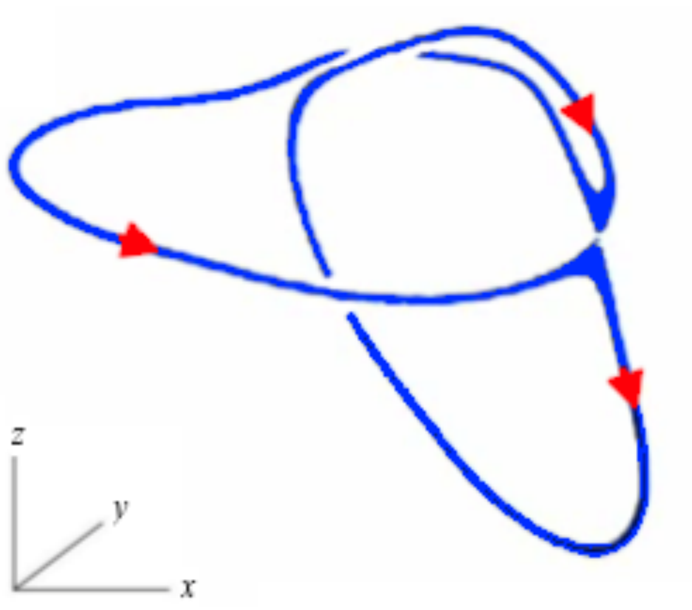


x-y plane sub-regions area and index			z-x plane sub-regions area and index			y-z plane sub-regions area and index		
R_1	201.35	+1	R_1	13.26	+1	R_1	199.48	+1
R_2	18.14	0	R_2	13.93	–1	R_2	22.85	0
R_3	0.94	+1	R_3	3.78	0	R_3	0.34	–1
R_4	0.39	–1	R_4	0.83	–1	R_4	0.86	+1
			R_5	1.70	+1	R_5	0.00	–1
			R_6	3.15	+1			
			R_7	2.18	–1			

+1	0	–1

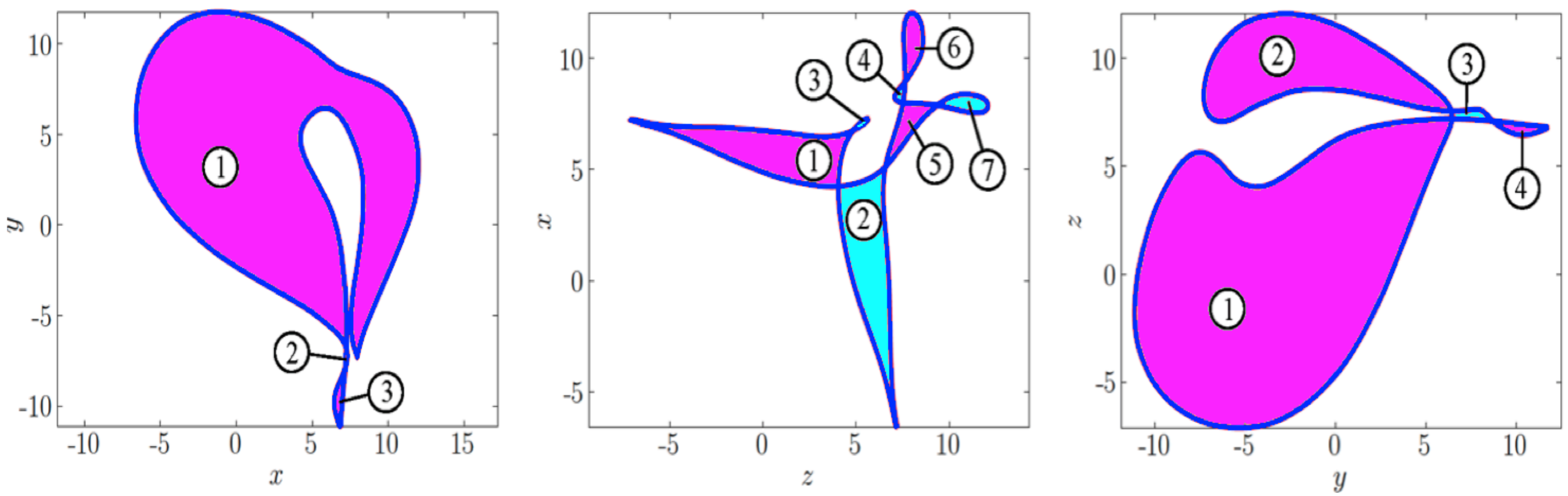


Weighted area computation: $t = 37$ *(Zucchher & Ricca PRE 2019)*

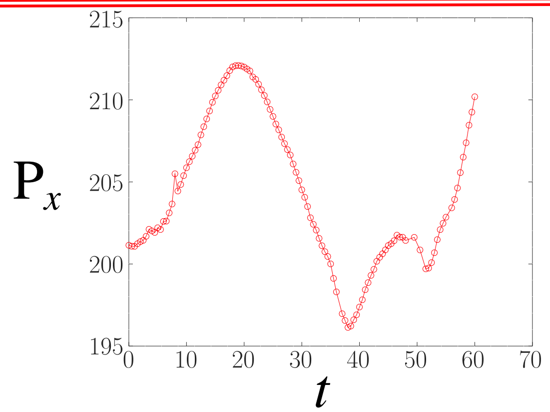


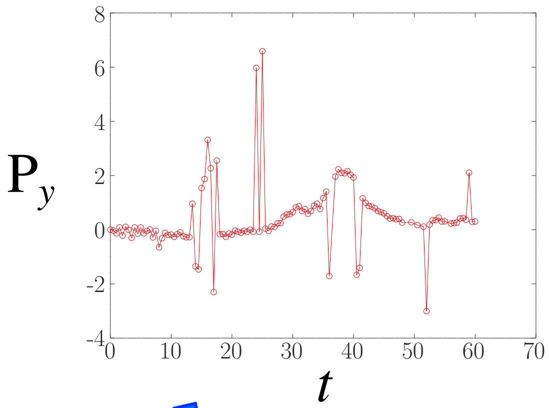
x-y plane sub-regions area and index			*z-x* plane sub-regions area and index			*y-z* plane sub-regions area and index		
R_1	201.77	+1	R_1	14.81	+1	R_1	159.18	+1
R_2	0.057	–1	R_2	15.63	–1	R_2	37.58	+1
R_3	1.0419	+1	R_3	0.10	–1	R_3	0.80	–1
			R_4	0.29	–1	R_4	1.00	+1
			R_5	2.44	+1			
			R_6	2.25	+1			
			R_7	1.54	–1			

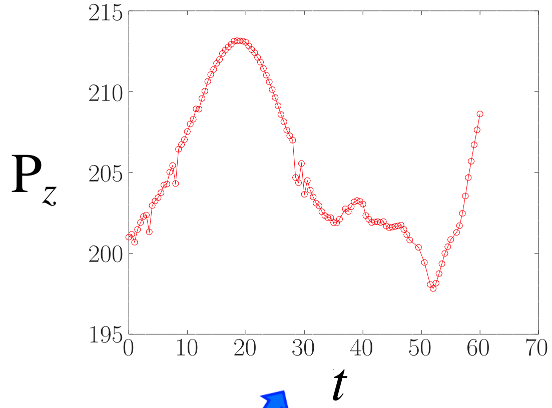
+1	0	–1



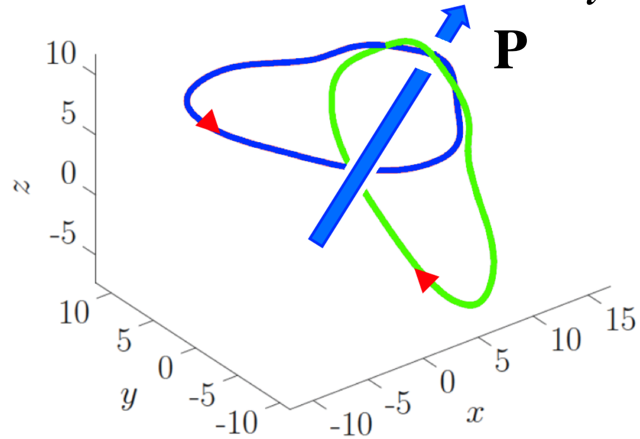
Resultant momentum of Hopf link and reconnecting rings

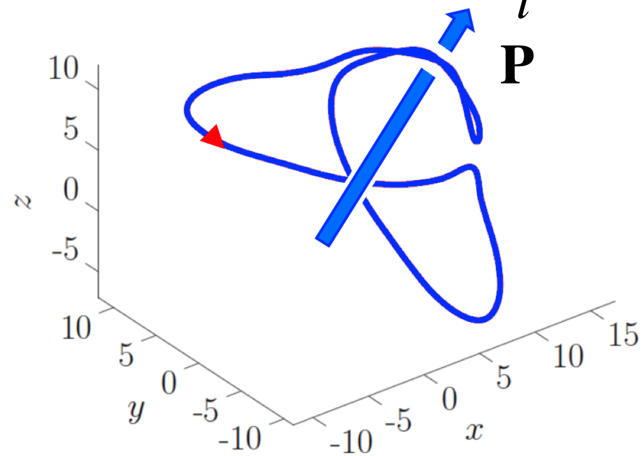




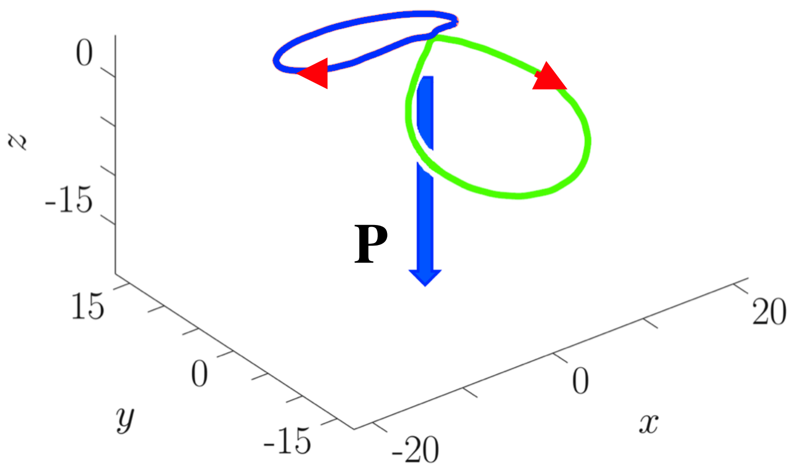


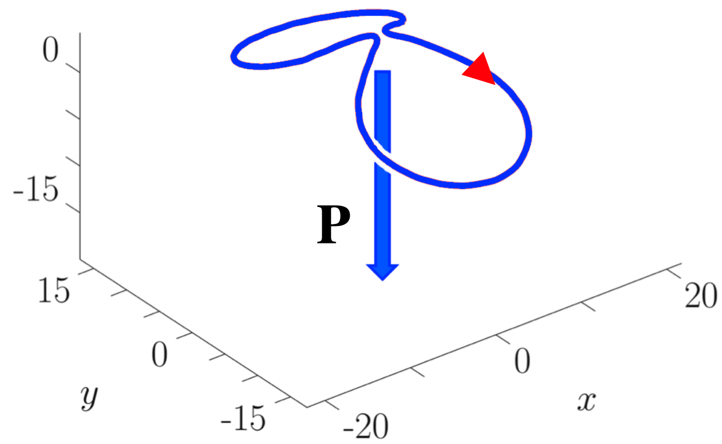
Hopf link





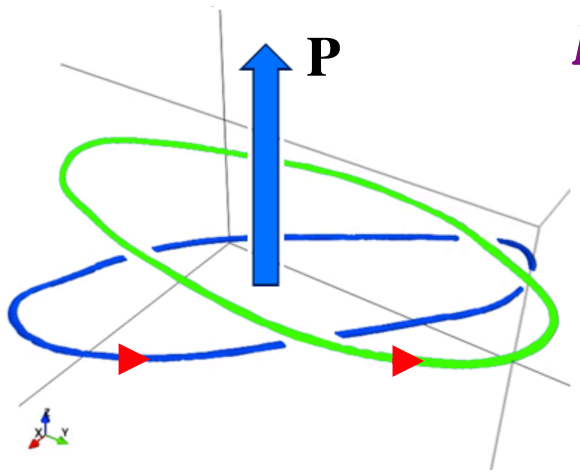
reconnecting rings



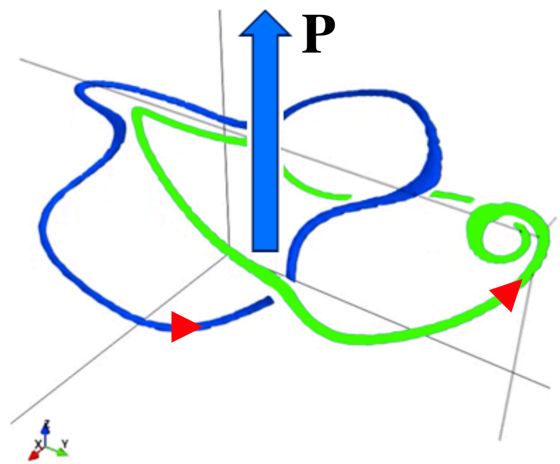


Production of Hopf link and trefoil knot from unlinked loops

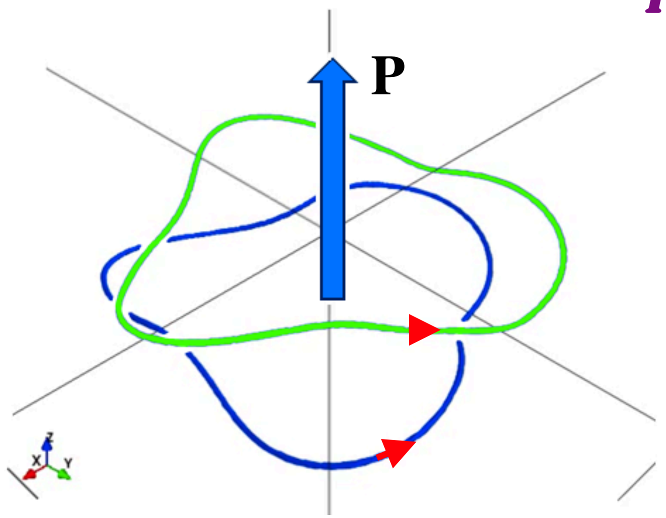
Hopf link



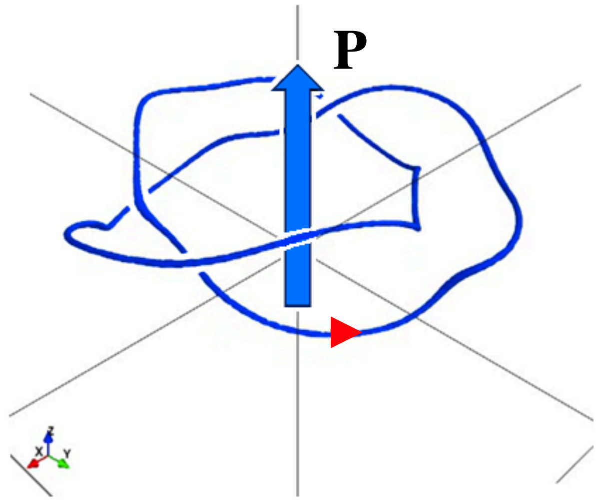
time



Trefoil knot



time



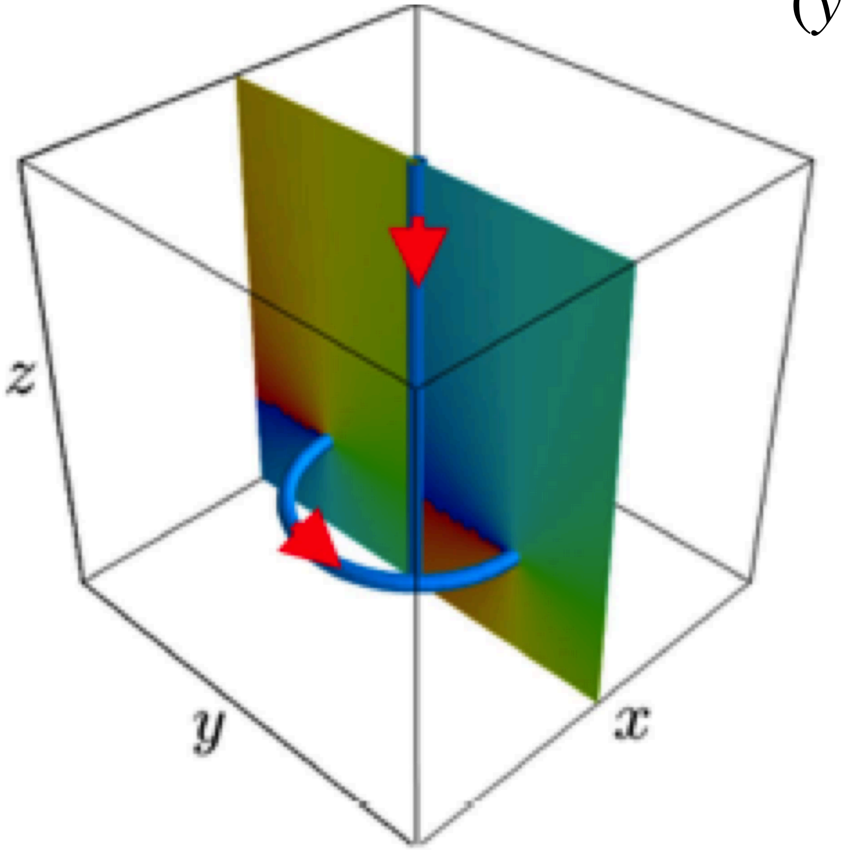
see movie

(Zuccher & Ricca 2019, to be submitted)

Physical effects of phase twist (Zuccher & Ricca FDR 2018)

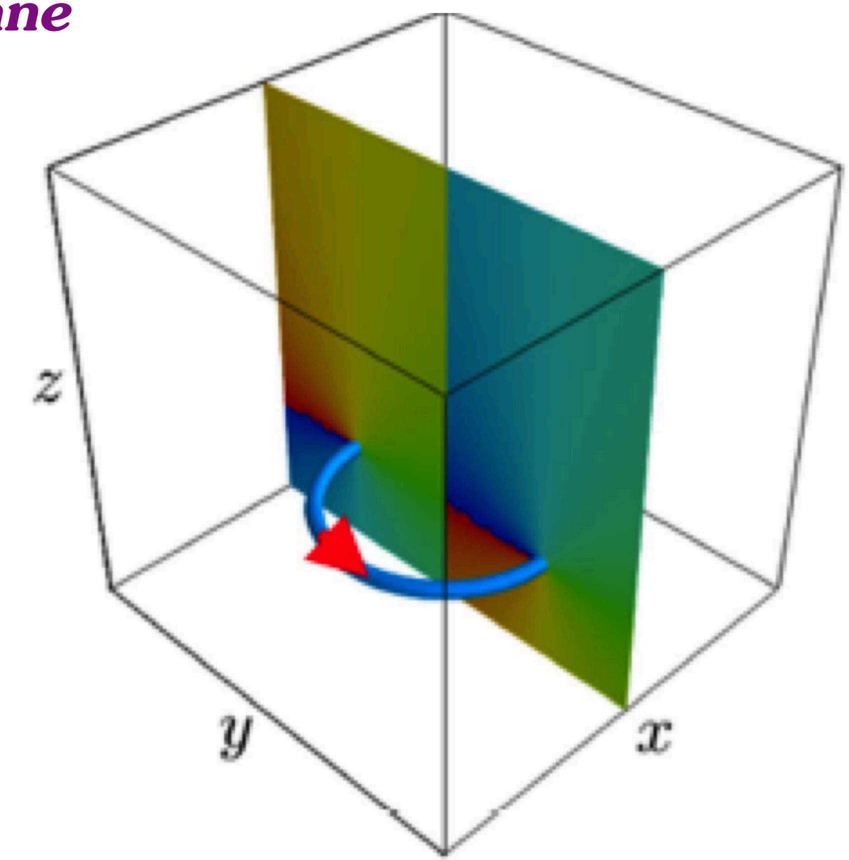
- ***Case A: twist induction***
- ***Case B: twist superposition***

phase contour in the (y-z) plane



$t = 0$

induction of phase twist $Tw = 1$ on vortex ring

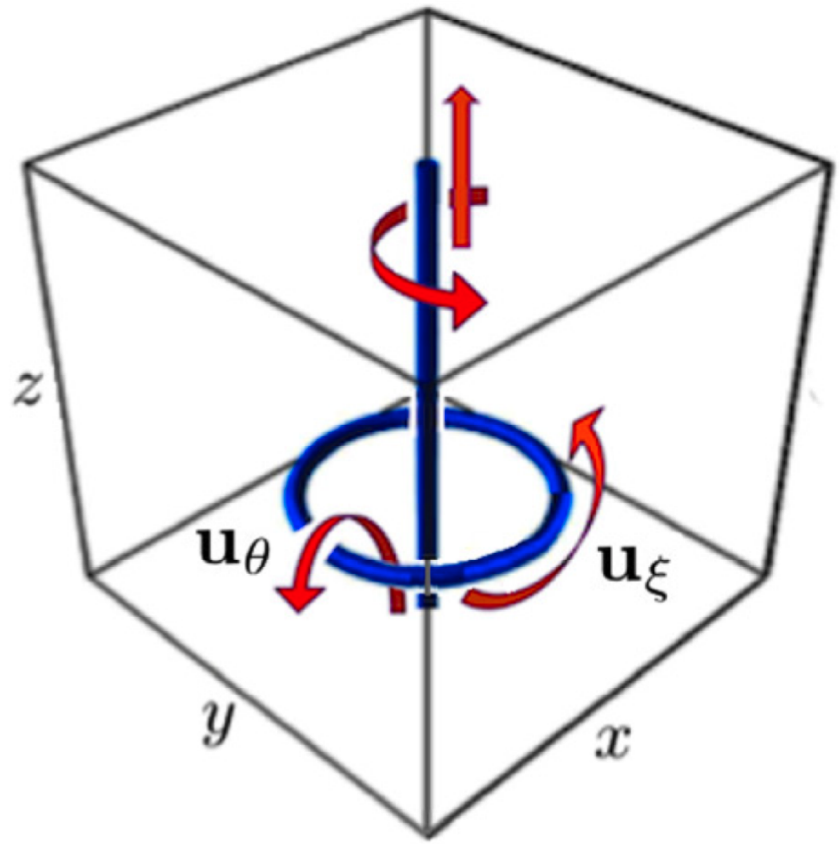


$t = 0$

superposition of phase twist $Tw = 1$ on vortex ring

Case A: twist induction

induction of phase twist $Tw = 1$ *on vortex ring*



$t = 0$

- ***Biot-Savart induction law:***

$$\mathbf{u}(\mathbf{x}) = \frac{\Gamma}{4\pi} \oint_{\mathcal{L}} \frac{\hat{\mathbf{T}}(\mathbf{X}^*) \times (\mathbf{x} - \mathbf{X}^*)}{|\mathbf{x} - \mathbf{X}^*|^3}\, ds$$

$$|\mathbf{u}_\xi| = \frac{\Gamma}{2\pi(R - a)}$$

$$\frac{|\mathbf{u}_\xi|}{U} = \frac{2}{\left(1 - \frac{a}{R}\right)\left(\ln \frac{8R}{a} - \frac{1}{2}\right)} \approx 0.49$$

Case B: twist superposition

- ***Theorem (Foresti & Ricca 2019).*** *Let $\mathcal{L}_1$ be a vortex ring of $\Gamma_1 = 1$. A rectilinear, central vortex $\mathcal{L}_2$ of $\Gamma_2 = 1$ can co-exists if and only if $\mathcal{L}_1$ and $\mathcal{L}_2$ are linked so that $Tw_1 + Tw_2 = \pm 2$.*

Proof.

(i) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are linked $\Longrightarrow$ $Tw_1 + Tw_2 = \pm 2$:

since $\Gamma_1 = \Gamma_2 = 1$, $H = 0 \Longrightarrow Lk_{\text{tot}} = 0$

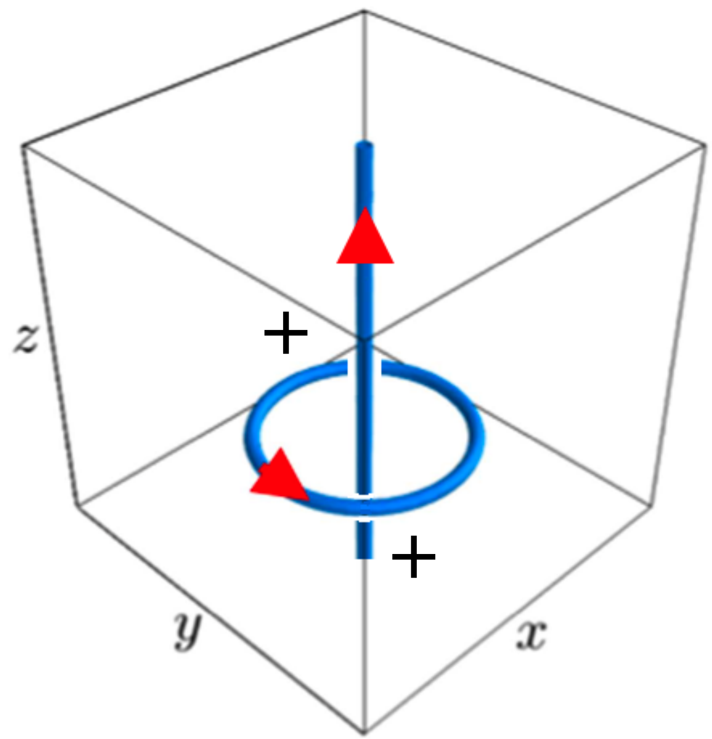
$$0 = 2Lk_{12} + (Wr_1 + Tw_1) + (Wr_2 + Tw_2)$$

$Wr_1 = 0$, $Wr_2 = 0$;

$$0 = 2Lk_{12} + Tw_1 + Tw_2$$

$$\Longrightarrow Tw_1 + Tw_2 = \pm 2 \; .$$

$Lk_{12} = +1$



We can prove that the lowest energy twist state is given by

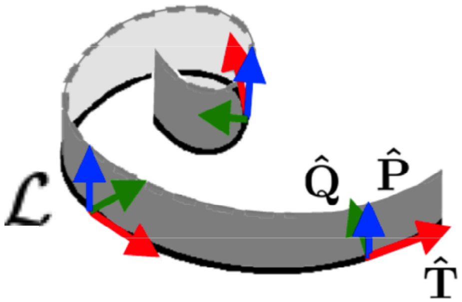
$$|Tw_1| = 1 \Longrightarrow |Tw_2| = 1 \, .$$

(ii) If there is $Tw_1 \Rightarrow \exists\, \mathcal{L}_2$ *such that* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are linked: suppose we have only* $\mathcal{L}_1 = \mathcal{L}$ *and for simplicity* $Tw_1 = Tw = 1$ *.*

- ***Twist.*** *The twist* Tw *of a unit vector* $\hat{\mathbf{P}}$ *on a curve* $\mathcal{L}$ *is defined by*

$$Tw = \frac{1}{2\pi}\oint_{\mathcal{L}} \left(\hat{\mathbf{P}} \times \frac{d\hat{\mathbf{P}}}{ds}\right)\cdot \hat{\mathbf{T}}\, ds \; .$$

- ***Zero-twist condition.*** *The unit vector* $\hat{\mathbf{P}}$ *does not rotate along* $\mathcal{L}$ *if and only if it is Fermi-Walker (FW)-transported along* $\mathcal{L}$*, i.e.*

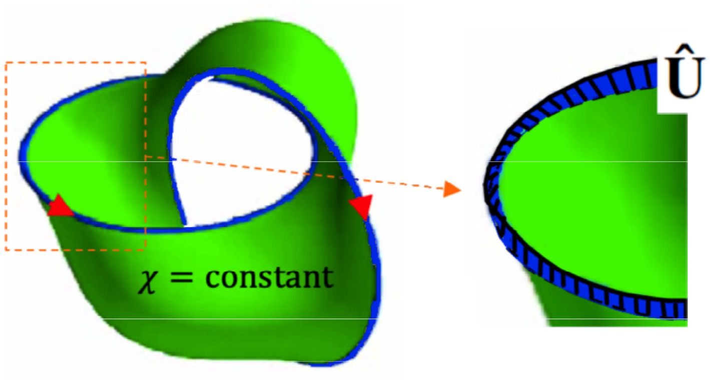


$$\frac{D_{\mathrm{FW}}\hat{\mathbf{P}}}{Ds} = \frac{d\hat{\mathbf{P}}}{ds} - (\hat{\mathbf{P}}\cdot\hat{\mathbf{T}})\frac{d\hat{\mathbf{T}}}{ds} + \left(\hat{\mathbf{P}}\cdot\frac{d\hat{\mathbf{T}}}{ds}\right)\hat{\mathbf{T}} = 0 \;, \forall s \in \mathcal{L} \;.$$

- ***Phase-twist.*** *Let* $\hat{\mathbf{U}}$ *be the ribbon unit vector on the isophase* $\chi =$ *cst.:*

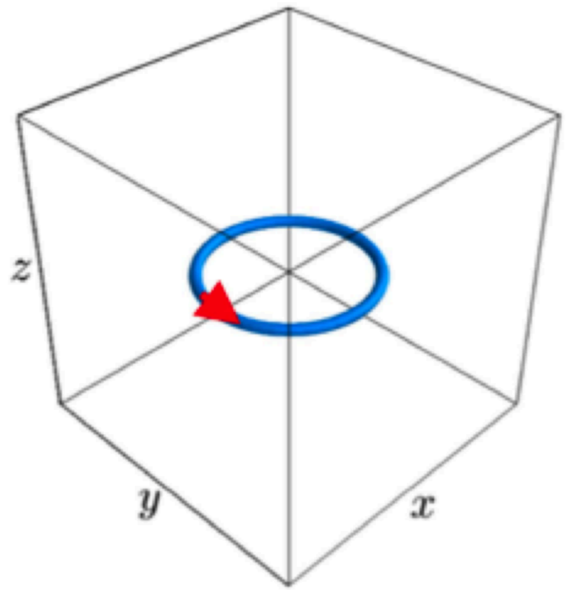
$$\frac{D_{\mathrm{FW}}\hat{\mathbf{U}}}{Ds} = \mathbf{\Omega}_\xi \times \hat{\mathbf{U}} = \Omega(\hat{\mathbf{T}}\times\hat{\mathbf{U}}) \;;$$

$$\frac{d\hat{\mathbf{U}}}{ds} = \frac{2\pi m}{L}\hat{\mathbf{e}}_\theta + c(\hat{\mathbf{B}}\times\hat{\mathbf{U}}) = \underbrace{\frac{2\pi m}{L}\hat{\mathbf{e}}_\theta}_{\mathbf{u}_\theta} \underbrace{- c\cos\theta\,\hat{\mathbf{T}}}_{\mathbf{u}_\xi} \;.$$



■

Twist injection by phase perturbation



- $Tw = 0$: ***dispersion relation for Kelvin waves***

$$\psi_0 \;\Rightarrow\; \psi = \psi_0 + \psi_1 + \ldots\,,$$

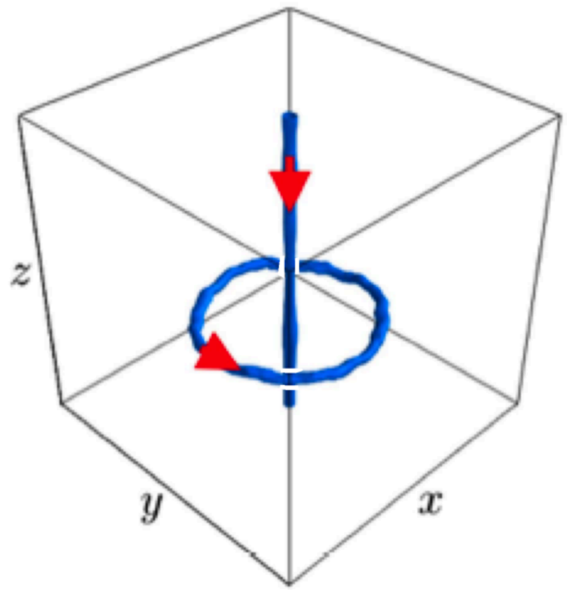
$$\psi_1 = \lambda e^{i(\boldsymbol{k}\cdot\boldsymbol{R}-\nu t)}\,, \quad |\lambda| \ll 1\,.$$

$$\Rightarrow \quad \nu = \frac{1}{2}(k^2 - 1) \quad \Rightarrow \quad \nabla\nu \propto k = |\mathbf{k}|$$

- $Tw \neq 0$: ***dispersion relation in presence of winding*** $w \in \mathbb{Z}$

$\psi = e^{iw\phi}\psi_0 + \lambda e^{i(\boldsymbol{k}\cdot\boldsymbol{R}-\nu t)}$ ***; after linearinzing we obtain***

$$\frac{\partial\psi_1}{\partial t} = \frac{i}{2}\left(\nabla + \frac{iw}{R}\hat{\mathbf{e}}_\phi\right)^2\psi_1 + \frac{i}{2}\psi_1\,,$$



with a new dispersion relation given by:

$$\boxed{\nu = \frac{1}{2}\left(\boldsymbol{k} + \frac{w}{R}\hat{\mathbf{e}}_\phi\right)^2 - \frac{1}{2}} \;\Rightarrow\; \nabla\nu \propto (k; w)\,.$$

(Foresti & Ricca, PRE 2019)