WCST2019

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Spontaneous stochasticity of shear-layer instabilities

by **Simon Thalabard**

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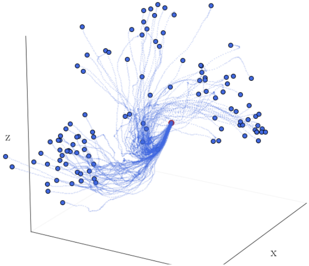
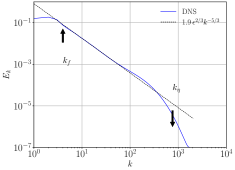
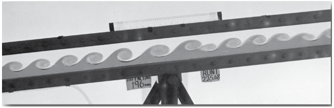
Joint work with **Alexei Mailybaev** *(IMPA)* and **Jérémie Bec** *(Mines Paris Tech)*

Turbulent dispersion

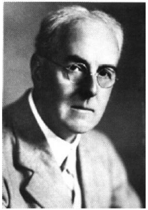




Shear layer instability



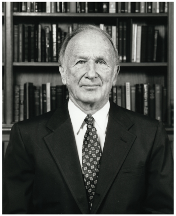
Intrinsic randomness of fluids



Atmospheric Diffusion shown on a Distance-Neighbour Graph.

By LEWIS F. RICHARDSON.

(Communicated by Sir Gilbert Walker, F.R.S.—Received November 7, 1925.)

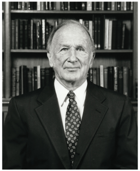


The predictability of a flow which possesses many scales of motion

By EDWARD N. LORENZ, *Massachusetts Institute of Technology*[1]

(Manuscript received October 31, 1968, revised version December 13, 1968)

Lorenz's 1969 conjecture



ABSTRACT

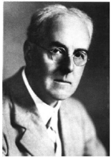
It is proposed that certain formally deterministic fluid systems which possess many scales of motion are observationally indistinguishable from indeterministic systems; specifically, that two states of the system differing initially by a small "observational error" will evolve into two states differing as greatly as randomly chosen states of the system within a finite time interval, which cannot be lengthened by reducing the amplitude of the initial error. The hypothesis is investigated with a simple mathematical model. An equation whose dependent variables are ensemble averages of the "error energy" in separate scales of motion is derived from the vorticity equation which governs two-dimensional incompressible flow. Solutions of the equation are determined by numerical integration, for cases where the horizontal extent and total energy of the system are comparable to those of the earth's atomsphere.

It is found that each scale of motion possesses an intrinsic finite range of predictability, provided that the total energy of the system does not fall off too rapidly with decreasing wave length. With the chosen values of the constants, "cumulus-scale" motions can be predicted about one hour, "synoptic-scale" motions a few days, and the largest scales a few weeks in advance. The applicability of the model to real physical systems, including the earth's atmosphere, is considered.

$\Rightarrow$ **Intrinsic finite-time randomness.**

Different from chaotic exponentiation, where finite-time errors can be made arbitrarily small

From Richardson's super-diffusion ...



> A general and beautiful theory of "Diffusion by Continuous Movements" has been given by G. I. Taylor.* It is expressed in terms of velocity.
>
> Although this theory of Taylor's is available, yet I think it will be a useful adventure to try now to make a theory of diffusion without assuming that $\Delta x/\Delta t$ has a limit.

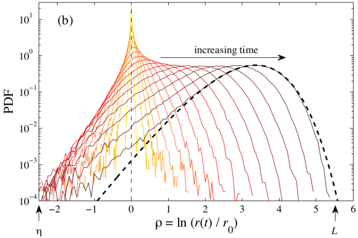
- *Multiplicative diffusion process:*

$$\partial_t p(r,t) = \partial_r \left(K(r) \partial_r p \right), \qquad K(r) \sim r^{4/3}$$

- *Expect self-similar asymptotics:*

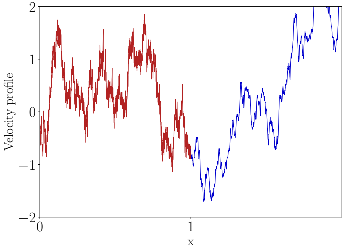
$$p(r,t) \to \frac{1}{r^\star(t)} \Psi\left(\frac{r}{r^\star(t)}\right), \qquad r^\star(t) \sim t^{3/2}$$

$\Rightarrow$ **statistical explosivity, *e.g.* "loss of memory"**



JFM 2015, with Krstulovic and Bec

... to **Spontaneous Stochasticity**



The modern view on Richardson's adventure

Advection in rough velocity ensembles:

$$\mathrm{d}\mathbf{X} = \mathrm{d}\mathfrak{v}_{\nu}(t, \mathbf{X}) + \sqrt{2\kappa}\,\mathrm{d}\mathbf{W}$$

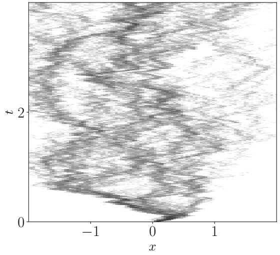
$$\delta\mathfrak{v}_{\nu} \sim r^{1/3} \text{ as } r, \nu \to 0$$

In the joint limit $\nu, \kappa \to 0$

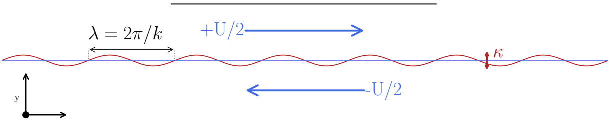
- $\mathbf{X}(\kappa, \nu)$ remains truly stochastic.
- Coincident trajectories may reach separations $O(1)$ in **finite time and almost surely.**

$\Longrightarrow$ They are **spontaneously stochastic.**

Bernard, Gawedzki, Kupiainen, Le Jan, Raimond,...



Kelvin-Helmholtz instability



Linear inviscid theory:

Exponential amplification with
growth rate $\sigma(k) = Uk/2$

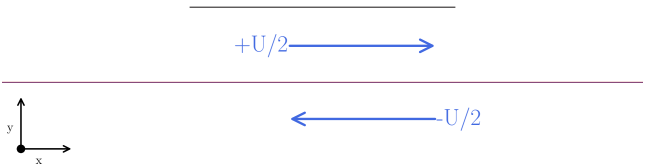
When the perturbation scale vanishes, *e.g.* $k \to \infty$,
the growth rate explodes: $\sigma(k) \to \infty$

$\Rightarrow$ **Breakdown of linear theory**

When the amplitude vanishes, the inviscid problem becomes ill-posed

$\Rightarrow$ **Singular initial-value problem**

Mathematical ambiguity of the inviscid case



- Initial-value problem is ill-posed
- Infinitely many dissipative solutions

Weak solutions to the incompressible Euler equations with vortex sheet initial data

László Székelyhidi, Jr.[a],
[a]*Hausdorff Center for Mathematics, University of Bonn*

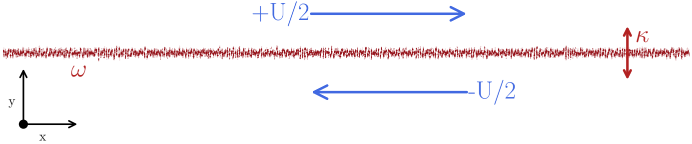
Abstract

We construct infinitely many admissible weak solutions to the incompressible Euler equations with initial data given by the classical vortex sheet. The construction is based on the method introduced recently in [2,3] using convex integration. In particular the vorticity is not a bounded measure. Instead, the energy decreases in time due to a linearly expanding turbulent zone around the vortex sheet.

$\Rightarrow$ **Can one predict the dynamics of a shear layer ?**

$\Rightarrow$ **Is the Kelvin-Helmholtz setup *spontaneously stochastic ?***

Unveiling spontaneous stochasticity



Physical regularization by white noise + viscosity

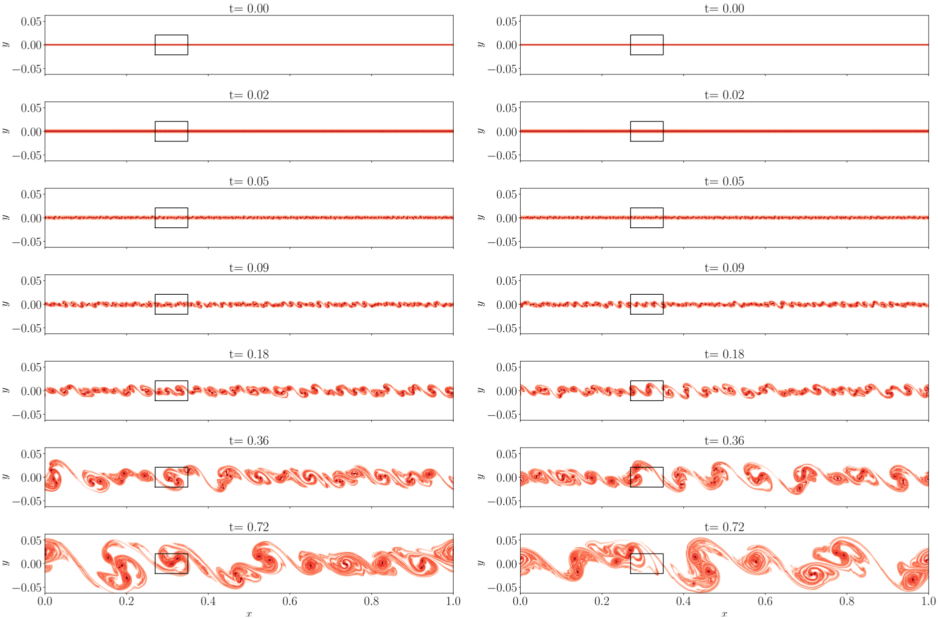
Navier-Stokes evolution:

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega \quad \text{with} \quad \mathbf{u} = -\nabla^{\perp} \left(\Delta^{-1} \omega \right)$$

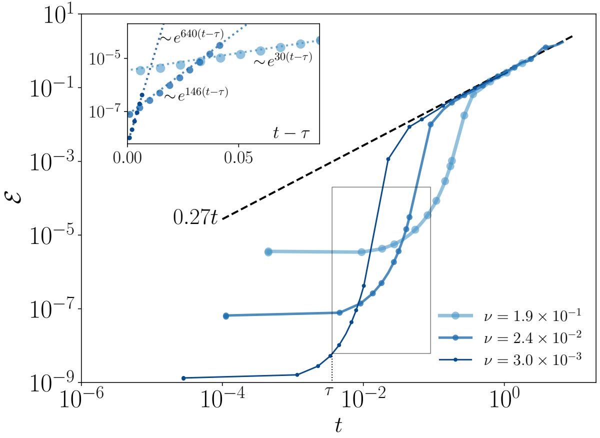
$$t = 0^+ : \quad \omega_\kappa(x, y) = U\delta(y) \, [1 + \kappa \eta(x)] \, , \qquad \text{with } \eta \text{ white noise}$$

Questions

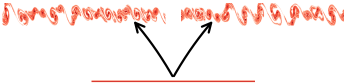
- Is there a non-trivial limit $\nu, \kappa \to 0$?
- Is it random?


t= 0.00
t= 0.02
t= 0.05
t= 0.09
t= 0.18
t= 0.36
t= 0.72
0.05
0.00
−0.05
y
0.0
0.2
0.4
0.6
0.8
1.0
x

Explosive separation of velocities

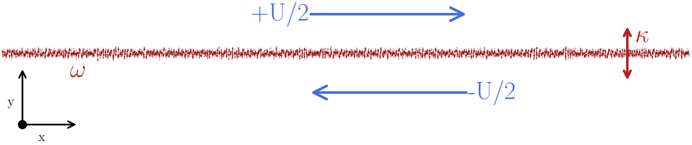


In the limit $\eta, \kappa \to 0$,
the dynamics is stochastic from $t = 0^+$



$\Rightarrow$ **It is *spontaneously* stochastic**

Universality of the dynamics



Two different regularizations

Navier-Stokes: $\nu, \kappa \to 0$

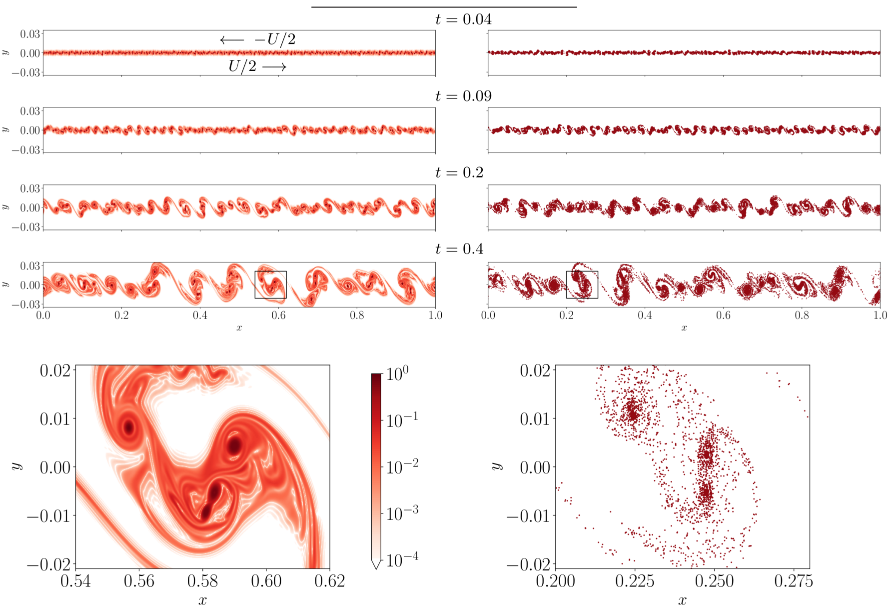
$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega \quad \text{with} \quad \mathbf{u} = -\nabla^{\perp} \left(\Delta^{-1} \omega \right)$$

Brikhoff-Rott: $N_b^{-1}, \kappa \to 0$

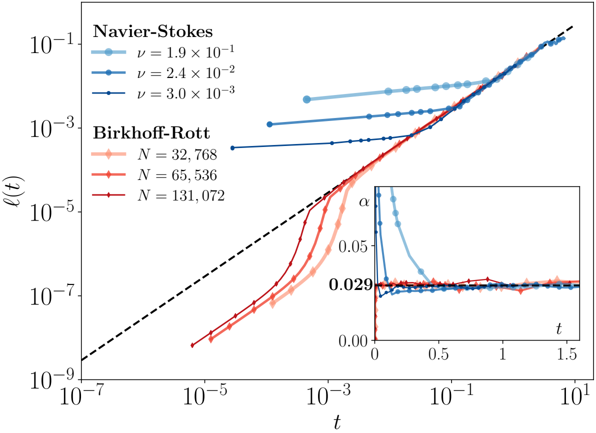
$$\dot{z}_n = \frac{1}{2i} \sum_{\substack{1 \le j \le N_b \\ j \ne n}} \Gamma_j \cot \left[\pi \left(z_n - z_j \right) \right] \qquad \Gamma_n = \frac{U}{N_b} (1 + \varepsilon \, N_b^{1/2} \, \eta_n)$$

Is the limiting random process dependent upon the regularization ?

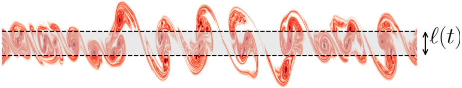
KH universality: *Navier-Stokes* vs *Birkhoff-Rott* regularization



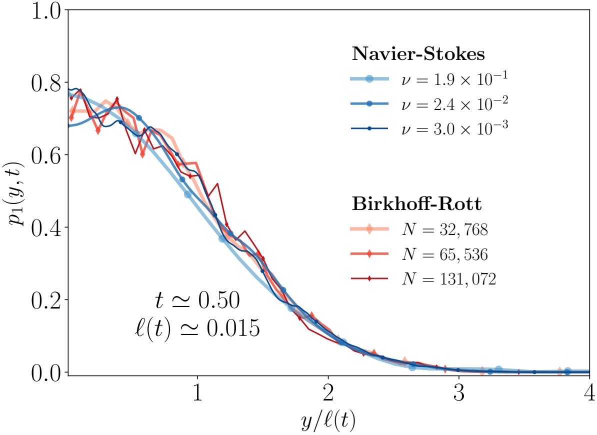
KH universality: Mixing length

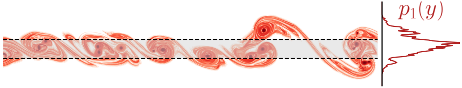


$$\ell = \alpha U t$$

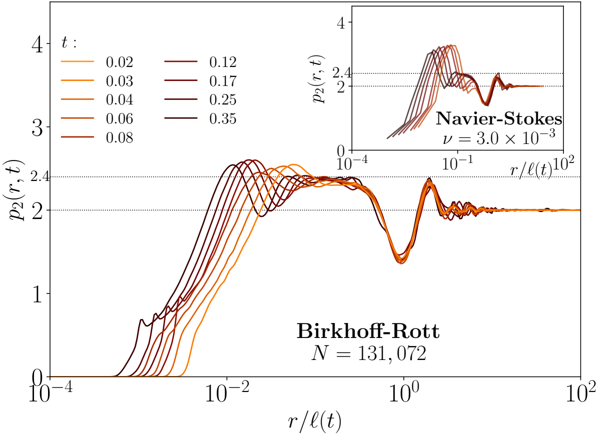


KH universality: Vorticity profile

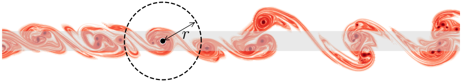




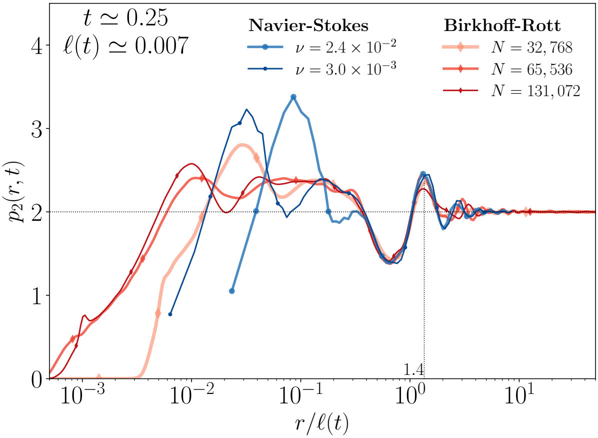
KH universality: Vorticity correlation



Statistical self-similarity

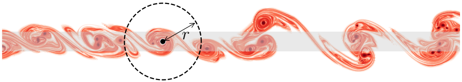


KH universality: Vorticity correlation



Statistical self-similarity

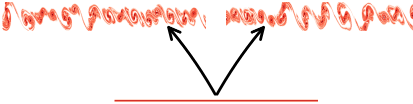
The limiting process is independent of the regularization



Summary of the observations:

KH instability is a physical example of a **spontaneously stochastic flow** :

- Deterministic at $t = 0$
- Random at $t > 0$

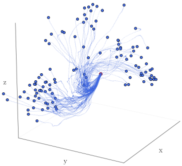


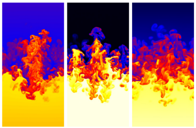
$\Rightarrow$ **More unpredictable than chaos**

The random process is

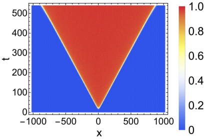
- Triggered by micro-scale fluctuations,
- Insensitive to those.

Where to chase spontaneous butterflies effects ?



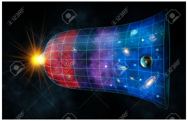


Biferale et al, 2018



Infinite spin chains

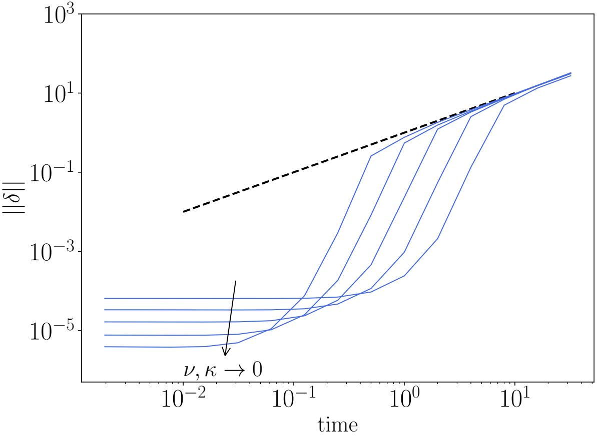
Das et al, 2018



Cosmology

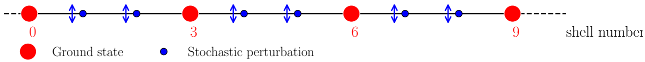
Thank you for your attention !

A toy model of an explosive KH instability



GOY/Sabra dynamics

“explosive” initial state



Inverse cascade of errors

How fast two *replica* of a multi-scale fluid system diverge ?

Assume

- $E(k) \sim k^{-e}$
- Local propagation $k \to k/2$ with timescale $\tau(k) \sim k^{(e-3)/2}$

Then, the error reaches $k = 1$ from ∞ at

$$T \sim \int_1^\infty \tau(k) d \log k \begin{cases} = \infty & \text{if } e \geq 3 \quad \text{smooth case, } \textit{e.g.}\text{, 2d direct cascade} \\ < \infty & \text{if } e < 3 \quad \text{rough case, } \textit{e.g.}\text{, 3d direct cascade} \end{cases}$$

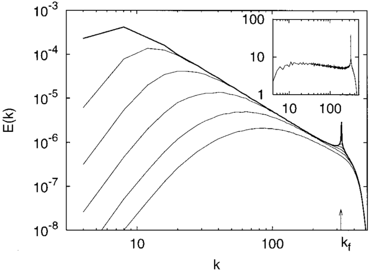


FIG. 1. Stationary energy spectrum $E(k)$ (thick line) and error spectrum $E_\Delta(k,t)$ at time $t=0.1,0.2,0.4,0.8,1.6$. $k_f=320$ is the forcing wavenumber. In the inset we plot the compensated spectrum $\epsilon^{-2/3}k^{5/3}E(k)$.

Boffetta & Musacchio, PoF 2001