



A note on an alternative derivation of the Benney equations for short wave–long wave interactions

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ARTICLE INFO

Article history:

Received 1 September 2011
Received in revised form
20 December 2011
Accepted 10 January 2012
Available online 18 January 2012

Keywords:

Water waves
Nonlinear waves
Benney equations
Short wave–long wave interaction
Zakharov equation
Nonlinear Schrödinger equation

ABSTRACT

Starting from the Zakharov formulation of surface gravity waves, we derive the short wave–long wave interaction equations. The procedure involves writing the original water wave problem in Fourier space in Hamiltonian form and expanding it in powers of wave steepness. The decomposition of long and short waves is then introduced in the evolution equation and a near identity transformation is used in order to remove the non-resonant terms. Some algebra is then needed to calculate the coefficients in the system of equations. The shallow water limit of such a system is also reported.

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1. Introduction

The interaction of short waves with long waves is an important problem in many physical systems. Examples may be found for internal gravity wave [1], capillary–gravity waves [2,3], internal waves with surface waves [4], sonic–Langmuir wave interactions in plasmas [5–7], optical–microwave interactions in nonlinear mediums [8], and bubbly liquids [9]. The understanding of the short wave–long wave interaction is also relevant in oceanography for applications to remote sensing techniques [10–12]. Concerning surface gravity waves, an important theoretical contributions to the subject was made in the 70s in [13]. In particular, Benney in 1976 generalized his study on the interaction between small and large scale surface waves, proposing a general theory for interactions between short and long waves [14]. In the limit that the amplitude of the long wave component is much smaller than the short wave one, the interaction equations, also known as the Benney equations [14], take the following form:

$$\begin{aligned} \frac{\partial L}{\partial t} + c_0 \frac{\partial L}{\partial x} &= \alpha \frac{\partial SS^*}{\partial x} \\ \frac{\partial S}{\partial t} + C_g \frac{\partial S}{\partial x} &= i \left(\beta \frac{\partial^2 S}{\partial x^2} + \delta LS + \gamma |S|^2 S \right). \end{aligned} \quad (1)$$

Here L and S represent the long and the short wave component respectively; c_0 , C_g , α , γ and δ are real constants which depend on the context in which the equations have been derived.

Strictly speaking, the resonance between the two wave components can be achieved only if the group velocity of the short wave is equal to the phase velocity of the long wave, i.e. $c_0 = C_g$. This is for example possible for capillary–gravity waves, [2], for internal waves [15] or for the interaction of internal waves with surface waves [16]. For pure gravity waves the group velocity of the short waves is always smaller than the phase velocity of the long waves and the exact resonance is never achieved. However, as the water depth decreases the group velocity tends to the phase velocity. In such a limit the above equations may play a relevant role also for pure gravity waves; see also [17].

The standard method for deriving Benney equations is the multiple scale expansion technique. The idea is to introduce slow independent variables (both for time and space) and treat each of them as independent. The extra degrees of freedom arising from such variables allows one to remove the secular terms that may appear in the standard expansion. The multiple scale expansion is usually performed in physical space and a simplification of the procedure is the requirement that the short waves are quasi-monochromatic.

In this paper, we present a completely different approach. Our starting point is the Zakharov formulation of surface gravity waves, where water wave equations are written using the Hamiltonian formalism [18]. The Hamiltonian is expanded in powers of wave amplitude (steepness) and written in Fourier space. The decomposition of the wave field into long and

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short waves is then inserted in the evolution equation and, via a near identity transformation, the non-resonant terms are systematically removed from the original equations. Some algebra is then needed for the calculations of the coefficients in the equations in the desired limits.

The paper is organized as follows. In Section 2, the Hamiltonian formulation of surface gravity waves is briefly described. In Section 3, the details of the derivation of the Benney equations and the methodology used is explained. In Section 4, the equations are presented and the shallow water limit is taken. In order to check the correctness of the coefficients in the equations, in Section 5 we briefly touch the derivation of the Nonlinear Schrödinger equation (NLS) in arbitrary depth from the Benney system and verify that the coefficient in front of the nonlinear term is exactly the same as the one originally derived in [19] using the standard multiple scale expansion.

2. The Hamiltonian formulation of surface gravity waves

The Zakharov's description of surface gravity waves is based on the Hamiltonian formulation. Here, without going into the details, we report the results described in [20]. The Hamiltonian, or the total energy of the system, is the sum of the potential plus the kinetic energy. The surface elevation, $\eta(x, y, t)$ and the velocity potential, $\psi(x, y, t)$, calculated on the surface are the canonical variables; here x and y are the horizontal coordinates and t is time. In the formulation introduced by Zakharov, see [18], variables are considered in Fourier space; the complex amplitude $a_k = a(\mathbf{k}, t)$ is introduced and related to the surface elevation and velocity potential as follows:

$$\begin{aligned} \eta(\mathbf{k}, t) &= \mathcal{M}(k)[a(\mathbf{k}, t) + a^*(-\mathbf{k}, t)], \quad \text{with} \\ \mathcal{M}(k) &= \left[\frac{q(k)}{2\omega(k)} \right]^{1/2} \\ \psi(\mathbf{k}, t) &= -i\mathcal{N}(k)[a(\mathbf{k}, t) - a^*(-\mathbf{k}, t)], \quad \text{with} \\ \mathcal{N}(k) &= \left[\frac{\omega(k)}{2q(k)} \right]^{1/2} \end{aligned} \quad (2)$$

where

$$\omega(k) = \sqrt{gk \tanh(kh)}, \quad q(k) = k \tanh(kh). \quad (3)$$

\mathbf{k} are horizontal wave vectors, $k = |\mathbf{k}|$, h the depth and g the gravity acceleration.

The Hamiltonian is expanded in nonlinearity (the small parameter is the wave steepness), resulting in:

$$\begin{aligned} H &= \int \omega_0 a_0^* a_0 dk_0 + \int U_{0,1,2}^{(1)} (a_0^* a_1 a_2 + a_0 a_1^* a_2^*) \delta_{0-1-2} dk_{012} \\ &+ \frac{1}{3} \int U_{0,1,2}^{(3)} (a_0^* a_1^* a_2^* + a_0 a_1 a_2) \delta_{0+1+2} dk_{012} \\ &+ \frac{1}{2} \int V_{0,1,2,3}^{(2)} a_0^* a_1^* a_2 a_3 \delta_{0+1-2-3} dk_{0123} \\ &+ \int V_{0,1,2,3}^{(1)} (a_0^* a_1 a_2 a_3 + a_0 a_1^* a_2^* a_3^*) \delta_{0-1-2-3} dk_{0123} \\ &+ \frac{1}{4} \int V_{0,1,2,3}^{(4)} (a_0^* a_1^* a_2^* a_3^* + a_0 a_1 a_2 a_3) \delta_{0+1+2+3} dk_{0123} \\ &+ \dots \end{aligned} \quad (4)$$

The Hamiltonian in (4) is truncated to third order in nonlinearity and “...” indicates five-wave (or more) interaction terms that have been neglected in the present derivation. In the above expressions the compact notation in which the argument in a , ω , $U^{(n)}$, $V^{(n)}$ and δ -functions have been replaced by subscripts. The

integral signs denote multiple integrals with limits $-\infty$ to $+\infty$ and the notation $dk_{012} = d\mathbf{k}_0 d\mathbf{k}_1 d\mathbf{k}_2$.

In the present paper we report only the analytical form of the coupling coefficients necessary for the derivation of the Benney equations (see Appendix), the others can be directly found in [20]. The evolution equation for the complex amplitude is given by:

$$\begin{aligned} i \frac{\partial a_0}{\partial t} &= \frac{\delta H}{\delta a_0^*} = \omega_0 a_0 + \int U_{0,1,2}^{(1)} a_1 a_2 \delta_{0-1-2} dk_{12} \\ &+ 2 \int U_{2,1,0}^{(1)} a_1^* a_2 \delta_{0+1-2} dk_{12} \\ &+ \int U_{0,1,2}^{(3)} a_1^* a_2^* \delta_{0+1+2} dk_{12} \\ &+ \int V_{0,1,2,3}^{(2)} a_1^* a_2 a_3 \delta_{0+1-2-3} dk_{123} \\ &+ \int V_{0,1,2,3}^{(1)} a_1 a_2 a_3 \delta_{0-1-2-3} dk_{123} \\ &+ 3 \int V_{3,2,1,0}^{(1)} a_1^* a_2^* a_3 \delta_{0+1+2-3} dk_{123} \\ &+ \int V_{0,1,2,3}^{(4)} a_1^* a_2^* a_3^* \delta_{0+1+2+3} dk_{123} + \dots \end{aligned} \quad (5)$$

where $\delta H / \delta a_0^*$ is a functional derivative.

It is important to note that in the Hamiltonian not all terms are equally important in the dynamics. It should be recalled that for surface gravity waves (neglecting surface tension), because of the shape of the dispersion relation, three wave interactions are never resonant and only some type of four wave interactions are resonant. For example the last three integrals in (5) do not contribute to the long term dynamics because the resonant manifold individuated by the following equations

$$\begin{aligned} \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 \pm \mathbf{k}_4 &= 0 \\ \omega_1 + \omega_2 + \omega_3 \pm \omega_4 &= 0 \end{aligned} \quad (6)$$

is empty.

3. Interaction of long and short waves

In order to derive a set of equations for the short wave–long wave interaction we split the complex amplitude wave field into two fields which corresponds to separate wave numbers in Fourier space:

$$a(\mathbf{k}, t) = L(\mathbf{k}, t) + S(\mathbf{k}, t) \quad (7)$$

where $L(\mathbf{k}, t)$ represent long waves traveling in shallow water while $S(\mathbf{k}, t)$ corresponds to waves in intermediate or deep water (short waves). We will make the hypothesis that $L \ll S$, i.e., the amplitude of the long waves is smaller than the amplitude of the short waves. Just to give some indicative numbers, we consider $L_k = L(\mathbf{k}, t)$ as a wave amplitude corresponding to waves whose non-dimensional wave number satisfies the following relation $kh \ll 1$; $S_k = S(\mathbf{k}, t)$ corresponds to waves characterized by $kh \simeq 1$. In other words, if k_L and k_S are the characteristic wave numbers long and short waves then we assume that $k_L \ll k_S$. Under this assumptions, the following rules will be used in the derivation of the model:

- (i) The short waves are quasi-monochromatic.
- (ii) Long waves have a smaller amplitude with respect to short waves ($L \ll S$), therefore terms of the type SS will dominate with respect to LL or LS ones; both the terms LL and LS are neglected in the long wave equation where the highest nonlinearity is given by SS terms. In the short wave equation, the cubic nonlinearity SSS is retained and the terms LL and LS are included in the equation.

- (iii) The interaction of two or more long waves cannot generate a short wave; indeed, the long waves are characterized by small wave numbers, therefore, because of the separation of the scales between the short and the long waves, the sum (or difference) of two (or more) small wave numbers (long waves) will not result in large wave number (short waves).
- (iv) The interaction of two short waves can be responsible for the generation of a long wave; indeed, short waves are characterized by large wave numbers, whose difference can give a small wave number (long wave).
- (v) The interaction of one or two long waves with a short wave will always result in a short wave. This is again due to the separation of scales between long and short waves (see iii).

3.1. Equation for the short waves

When the short wave–long wave decomposition is inserted in the evolution Eq. (5) the following equation is obtained for the short waves:

$$\begin{aligned}
 i \frac{\partial S_0}{\partial t} - \omega_{0S} S_0 = & 2 \int U_{2S,1S,0S}^{(1)} S_1^* S_2 \delta_{0+1-2} dk_{12} \\
 & + \int U_{0S,1S,2S}^{(1)} S_1 S_2 \delta_{0-1-2} dk_{12} \\
 & + \int U_{0S,1S,2S}^{(3)} S_1^* S_2^* \delta_{0+1+2} dk_{12} \\
 & + \int U_{0S,1L,2S}^{(3)} L_1^* S_2^* \delta_{0+1+2} dk_{12} \\
 & + 2 \int U_{2L,1S,0S}^{(1)} S_1^* L_2 \delta_{0+1-2} dk_{12} \\
 & + 2 \int U_{0S,1L,2S}^{(1)} L_1 S_2 \delta_{0-1-2} dk_{12} \\
 & + 2 \int U_{2S,1L,0S}^{(1)} L_1^* S_2 \delta_{0+1-2} dk_{12} \\
 & + \int V_{0S,1S,2S,3S}^{(2)} S_1^* S_2 S_3 \delta_{0+1-2-3} dk_{123} \\
 & + 3 \int V_{3S,2S,1S,0S}^{(1)} S_1^* S_2^* S_3 \delta_{0+1+2-3} dk_{123} \\
 & + \int V_{0S,1S,2S,3S}^{(4)} S_1^* S_2^* S_3^* \delta_{0+1+2+3} dk_{123} \\
 & + \int V_{0S,1S,2S,3S}^{(1)} S_1 S_2 S_3 \delta_{0-1-2-3} dk_{123}. \quad (8)
 \end{aligned}$$

For the first five integrals the resonant condition cannot be satisfied; the sixth and the seventh integrals can in principle be quasi-resonant in the shallow water limit and are responsible for the coupling between the short and the long waves. The eighth integral is responsible for the self-interaction of short waves and the exact resonance condition can be satisfied. The terms that are never resonant can be removed with a near identity transformation of the form:

$$\begin{aligned}
 S_0 = & s_0 + \int \mathcal{A}_{0S,1S,2S}^{(1)} S_1 S_2 \delta_{0-1-2} dk_{12} \\
 & + \int \mathcal{A}_{0S,1S,2S}^{(2)} S_1^* S_2 \delta_{0+1-2} dk_{12} \\
 & + \int \mathcal{A}_{0S,1S,2S}^{(3)} S_1^* S_2^* \delta_{0+1+2} dk_{12} \\
 & + \int \mathcal{B}_{2L,1S,0S}^{(1)} S_1^* L_2 \delta_{0+1-2} dk_{12} \\
 & + \int \mathcal{B}_{0S,1L,2S}^{(3)} L_1^* S_2^* \delta_{0+1+2} dk_{12}
 \end{aligned}$$

$$\begin{aligned}
 & + \int \mathcal{D}_{0S,1S,2S,3S}^{(1)} S_1 S_2 S_3 \delta_{0-1-2-3} dk_{123} \\
 & + \int \mathcal{D}_{0S,1S,2S,3S}^{(2)} S_1^* S_2 S_3 \delta_{0+1-2-3} dk_{123} \\
 & + \int \mathcal{D}_{0S,1S,2S,3S}^{(3)} S_1^* S_2^* S_3 \delta_{0+1+2-3} dk_{123} \\
 & + \int \mathcal{D}_{0S,1S,2S,3S}^{(4)} S_1^* S_2^* S_3^* \delta_{0+1+2+3} dk_{123} + \dots \quad (9)
 \end{aligned}$$

The above transformation has been used in [20] to reduce the Hamiltonian of surface gravity waves to its canonical form. Indeed, it is an *ad hoc* transformation which has the role of removing the non-resonant terms in the equation of motion. The transformation (9) is inserted in the Eq. (8) and the matrices \mathcal{A} , \mathcal{B} , \mathcal{D} are determined in order to remove the mentioned non-resonant terms. From a physical point of view, the role of the transformation is to remove from the equation of motion the so called bound wave components which do not play a role in the exchange of energy between modes. The resulting modes, s_k , are called free waves. When applied to a Stokes wave, the effect of the transformation is to remove all harmonics (see [21] for details).

After the transformation, the Eq. (8) simplifies notably and reads:

$$\begin{aligned}
 i \frac{\partial s_0}{\partial t} - \omega_{0S} s_0 = & 2 \int U_{0S,1L,2S}^{(1)} L_1 S_2 \delta_{0-1-2} dk_{12} \\
 & + 2 \int U_{2S,1L,0S}^{(1)} L_1^* S_2 \delta_{0+1-2} dk_{12} \\
 & + \int \tilde{V}_{0S,1S,2S,3S}^{(2)} S_1^* S_2 S_3 \delta_{0+1-2-3} dk_{123} \quad (10)
 \end{aligned}$$

where $\tilde{V}^{(2)}$, reported in the Appendix, results from the transformation used to removed the first three terms in the Eq. (8). Note that the effect of the near-identity transformation (9) has been to remove the first five and the last three integrals in Eq. (8). Such procedure has some consequences on the resulting four wave interaction term (last term) in Eq. (10): the coupling coefficient in the integral is now $\tilde{V}_{0S,1S,2S,3S}^{(2)}$, which is related to $V_{0S,1S,2S,3S}^{(2)}$ as expressed in the Appendix. Physically, this implies that three-wave non resonant interactions are now re-arranged as four wave resonant interactions. The procedure of removing integrals which contain terms of the form of $L^* S^*$ or $L S^*$ would give a contribution at a higher order with respect to the Eq. (10), therefore we have avoided the calculation of the coefficients $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(3)}$. The coefficients $\mathcal{D}^{(i)}$ have a lengthy expression which can be found in [20] and the $\mathcal{A}^{(i)}$ can be found in the Appendix.

We confine the dynamic in one horizontal dimension and introduce the following envelope variable:

$$A_k = \left(\frac{2q_k}{\omega_k} \right)^{1/2} s_k e^{i\omega_p t} \quad (11)$$

where $\omega_p = \omega(k_p)$ is the angular frequency corresponding to the dominant wave number, k_p , of the short waves. After making the change of variable $k = \chi + k_p$, with χ a small quantity with respect to k_p and Taylor expanding ω_p around k_p , Eq. (10) becomes:

$$\begin{aligned}
 i \frac{\partial A_0}{\partial t} - \frac{\partial \omega}{\partial k} \Big|_{k_p} \chi_0 A_0 - \frac{1}{2} \frac{\partial^2 \omega}{\partial k^2} \Big|_{k_p} \chi_0^2 A_0 \\
 = \frac{i}{2\pi} \int k_p k_1 \psi_{L1} A_2 \delta_{0-1-2} dk_{12} \\
 + \frac{k_p^2 g}{2\pi} \left(\frac{1 - \tanh[k_p h]^2}{2\omega_p} \right) \int \eta_{L1} A_2 \delta_{0-1-2} dk_{12} \\
 + \frac{\omega_p}{2q_p} \tilde{V}_{p,p,p,p}^{(2)} \int A_1^* A_2 A_3 \delta_{0+1-2-3} dk_{123}, \quad (12)
 \end{aligned}$$

with

$$\tilde{V}_{p,p,p,p}^{(2)} = \frac{k_p^3}{4\pi^2} \frac{9 \tanh[k_p h]^4 - 10 \tanh[k_p h]^2 + 9}{8 \tanh[k_p h]^3}. \quad (13)$$

η_L and ψ_L are related to L via Eq. (2). Using the following definition of the Fourier Transform

$$A(x, t) = \frac{1}{2\pi} \int A(\chi, t) \exp(i\chi x) d\chi, \quad (14)$$

we write Eq. (12) in physical space:

$$\begin{aligned} i \frac{\partial A}{\partial t} + i C_g \frac{\partial A}{\partial x} \\ = \beta \frac{\partial^2 A}{\partial x^2} + k_p \frac{\partial \psi_L}{\partial x} A + k_p^2 g \left(\frac{1 - \tanh[k_p h]^2}{2\omega_p} \right) \eta_L A \\ + 4\pi^2 \frac{\omega_p}{2q_p} \tilde{V}_{p,p,p,p}^{(2)} |A|^2 A \end{aligned} \quad (15)$$

with

$$C_g = \left. \frac{\partial \omega}{\partial k} \right|_{k_p}, \quad \beta = - \left. \frac{1}{2} \frac{\partial^2 \omega}{\partial k^2} \right|_{k_p}. \quad (16)$$

Thus, in order to express ψ_L as a function of η_L and A , we have first to derive the equation for the long waves.

3.2. Equation for the long waves

Inserting the decomposition (7) into Eq. (5) and making use of the aforementioned rules, it is straight forward to obtain the following equation:

$$\begin{aligned} i \frac{\partial L_0}{\partial t} - \omega_{0L} L_0 = 2 \int U_{2S,1S,0L}^{(1)} S_1^* S_2 \delta_{0+1-2} dk_{12} \\ + \int U_{0L,1S,2S}^{(1)} S_1 S_2 \delta_{0-1-2} dk_{12} \\ + \int U_{0L,1S,2S}^{(3)} S_1^* S_2^* \delta_{0+1+2} dk_{12}. \end{aligned} \quad (17)$$

This is an integro-differential equation which describes the linear evolution of long waves forced by short waves. The last two terms can be safely be neglected because the interaction of two short waves propagating in the same direction and satisfying the δ functions in the integrals cannot generate a long wave. The equation is written for variable L . For practical purposes it is custom to write the equation for the surface elevation η_L ; the procedure involves writing an evolution equation for L_{-k}^*

$$-i \frac{\partial L_{-0}^*}{\partial t} - \omega_{0L} L_{-0}^* = 2 \int U_{-2S,-1S,-0L}^{(1)} S_{-1} S_{-2}^* \delta_{0+1-2} dk_{12} \quad (18)$$

and subtract it from (17). In the long wave approximation the dispersion relation is expanded for small kh to the leading order, $\omega_k = \sqrt{gh}k$. Using the transformation (9) to the leading order and relation (11), the resulting equation for one dimensional propagation is the following:

$$\frac{\partial \eta_{L0}}{\partial t} - hk_0^2 \psi_{L0} = -i \frac{\omega_p k_p}{4\pi q_p} \int k_0 A_1^* A_2 \delta_{0+1-2} dk_{12}. \quad (19)$$

The variable ψ appears in the equation and it is necessary to write an evolution equation for it. Summing Eqs. (17) and (18) and following the procedure used for the derivation of the equation for η_L , after some algebra, we get:

$$\frac{\partial \psi_{L0}}{\partial t} + g \eta_{L0} = - \frac{g(k_p^2 - q_p^2)}{8\pi q_p} \int A_1^* A_2 \delta_{0+1-2} dk_{12}. \quad (20)$$

Using the definition of the Fourier transform in (14) we can write Eqs. (19) and (20) in real space:

$$\frac{\partial \eta_L}{\partial t} + h \frac{\partial^2 \psi_L}{\partial x^2} = - \frac{\omega_p k_p}{2q_p} \frac{\partial |A|^2}{\partial x} \quad (21)$$

$$\frac{\partial \psi_L}{\partial t} + g \eta_L = - \frac{g(k_p^2 - q_p^2)}{4q_p} |A|^2. \quad (22)$$

In order to derive an equation for waves propagating in a single direction (for example positive x), we find useful to take the derivative of Eq. (22) with respect to x and introduce the variable $u = \partial \psi_L / \partial x$. Eqs. (21) and (22) becomes:

$$\frac{\partial \eta_L}{\partial t} + h \frac{\partial u}{\partial x} = - \frac{\omega_p k_p}{2q_p} \frac{\partial |A|^2}{\partial x} \quad (23)$$

$$\frac{\partial u}{\partial t} + g \frac{\partial \eta_L}{\partial x} = - \frac{g(k_p^2 - q_p^2)}{4q_p} \frac{\partial |A|^2}{\partial x}. \quad (24)$$

We consider waves traveling to the right and introduce the following ansatz:

$$u = + \frac{c_0}{h} \eta_L + \mu |A|^2 \quad (25)$$

where $c_0 = \sqrt{gh}$ is the wave phase velocity in shallow water; the coefficient μ is found in such a way to make the Eqs. (23)–(24) match (the same procedure is used in Whitham book [22, p. 466], to derive the Korteweg–de Vries equation from the Boussinesq's equation). To the leading order, Eq. (15) reduces to:

$$\frac{\partial A}{\partial t} \simeq -C_g \frac{\partial A}{\partial x}, \quad (26)$$

therefore, the following coefficient is obtained:

$$\mu = \frac{1}{c_0 + C_g} \left(\frac{-2\omega_p k_p c_0 + gh(k_p^2 - q_p^2)}{4hq_p} \right). \quad (27)$$

The resulting equation for the surface elevation is:

$$\frac{\partial \eta_L}{\partial t} + c_0 \frac{\partial \eta_L}{\partial x} = \alpha \frac{\partial |A|^2}{\partial x} \quad (28)$$

with

$$\alpha = - \frac{k_p}{4} \frac{c_0^2}{c_0 + C_g} \left[\frac{2(1 - \tanh[k_p h]^2)}{\tanh[k_p h]} + \frac{1}{k_p h} \right]. \quad (29)$$

Eq. (28) represents the first equation in Benney system.

4. The Benney equations

We now reconsider the equation for the short waves; recalling that $\partial \psi / \partial x = u$ and inserting Eq. (25) in (15), we finally arrive to the second equation in the Benney system (see 1):

$$i \frac{\partial A}{\partial t} + i C_g \frac{\partial A}{\partial x} = \left[\beta \frac{\partial^2 A}{\partial x^2} + \delta \eta_L A + \gamma |A|^2 A \right] \quad (30)$$

where

$$\delta = k_p^2 \left(\frac{c_0}{k_p h} + \frac{g(1 - \tanh[k_p h]^2)}{2\omega_p} \right), \quad (31)$$

$$\begin{aligned} \gamma = 4\pi^2 \tilde{V}_{p,p,p,p}^{(2)} \left(\frac{\omega_p}{2k_p \tanh[k_p h]} \right) \\ + \frac{k_p^2 c_0}{c_0 + C_g} \left[\frac{c_0 k_p (1 - \tanh[k_p h]^2) - 2\omega_p}{4k_p h \tanh[k_p h]} \right]. \end{aligned} \quad (32)$$

Eqs. (28) and (30) are the Benney equations.

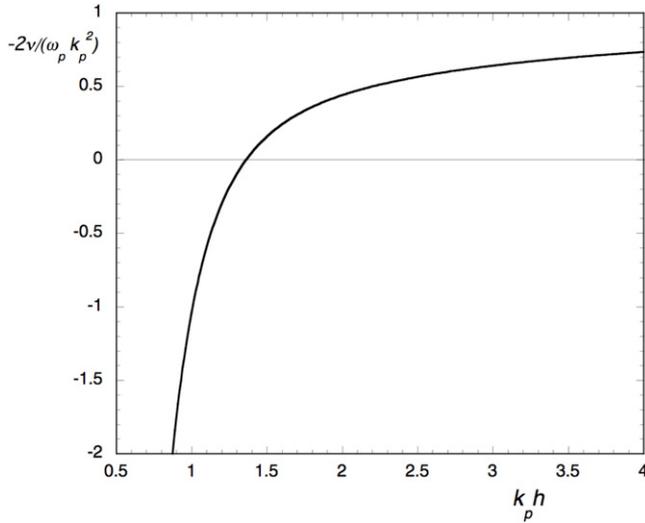


Fig. 1. Coefficient of the NLS equation as a function of $k_p h$ normalized by its value for infinite depth.

4.1. The shallow water limit of the Benney equations

In the shallow water limit the group velocity and the phase velocity are the same. Taking the limit for $k_p h \rightarrow 0$ and considering a coordinate system moving at $C_g = c_0$, the following equations are recovered:

$$\frac{\partial \eta_L}{\partial t} = -\frac{3 c_0}{8 h} \frac{\partial |A|^2}{\partial x} \quad (33)$$

$$i \frac{\partial A}{\partial t} = \frac{k_p h^2 c_0}{2} \frac{\partial^2 A}{\partial x^2} + \frac{c_0 k_p}{h} \eta_L A + \frac{9}{16} \frac{c_0}{h^4 k_p} |A|^2 A. \quad (34)$$

5. Verification of the coefficients in the Benney equations

In order to verify that the coefficients are correct, we derive the Nonlinear Schrödinger equation and check them against the equation derived in [19]. Starting from

$$\frac{\partial \eta_L}{\partial t} + c_0 \frac{\partial \eta_L}{\partial x} = \alpha \frac{\partial |A|^2}{\partial x} \quad (35)$$

$$i \frac{\partial A}{\partial t} + i C_g \frac{\partial A}{\partial x} = \beta \frac{\partial^2 A}{\partial x^2} + \delta \eta_L A + \gamma |A|^2 A \quad (36)$$

and following the procedure described in pp. 89–90 in [19] which basically involves introducing a slow time scale ($T = \epsilon^2 t$) and selecting a coordinate system moving with the group velocity C_g , we arrive to the following Nonlinear Schrödinger equation:

$$i \frac{\partial A}{\partial t} = \beta \frac{\partial^2 A}{\partial x^2} + \nu |A|^2 A \quad (37)$$

with

$$\nu = \gamma + \frac{\alpha \delta}{c_0 - C_g}. \quad (38)$$

The coefficient in front of the nonlinear term has been found to be exactly the same as the one derived in [19], see also [23], in Fig. 1 we show the coefficient as a function of kh : the coefficient change sign for $k_p h \simeq 1.36$, a well known result in the modulation theory for water waves.

6. Conclusions

In the present paper we have presented an alternative derivation of short wave–long wave interaction equations. Instead of using the multiple scale expansion and working in physical space, the derivation is performed in Fourier space and the non resonant terms are removed using a near identity transformation. The present method can be easily extended to derive the interaction equations in two horizontal dimensions and at higher order both in nonlinearity and dispersion. The Benney equations are derived including all the coefficients for the water wave problem. Their shallow water limit is also presented. The correctness of our procedure is assured by the fact that for long time scale the nonlinear Schrödinger equation for arbitrary depth is recovered.

Acknowledgments

M.O. acknowledges ONR grant N000141010991. The author thanks Al Osborne for discussions during the early stages of the work.

Appendix

Here we report the analytical form of a number of coupling coefficients:

$$U_{1,2,3}^{(1)} = \frac{1}{8\pi\sqrt{2}} \left\{ [\mathbf{k}_1 \cdot \mathbf{k}_2 - q_1 q_2] \left(\frac{g\omega_3}{\omega_1 \omega_2} \right)^{1/2} + [\mathbf{k}_1 \cdot \mathbf{k}_3 - q_1 q_3] \left(\frac{g\omega_2}{\omega_1 \omega_3} \right)^{1/2} + [\mathbf{k}_2 \cdot \mathbf{k}_3 + q_2 q_3] \left(\frac{g\omega_1}{\omega_2 \omega_3} \right)^{1/2} \right\},$$

$$U_{1,2,3}^{(3)} = \frac{1}{8\pi\sqrt{2}} \left\{ [\mathbf{k}_1 \cdot \mathbf{k}_2 + q_1 q_2] \left(\frac{g\omega_3}{\omega_1 \omega_2} \right)^{1/2} + [\mathbf{k}_1 \cdot \mathbf{k}_3 + q_1 q_3] \left(\frac{g\omega_2}{\omega_1 \omega_3} \right)^{1/2} + [\mathbf{k}_2 \cdot \mathbf{k}_3 + q_2 q_3] \left(\frac{g\omega_1}{\omega_2 \omega_3} \right)^{1/2} \right\},$$

$$\begin{aligned} \tilde{V}_{1,2,3,4}^{(2)} &= V_{1,2,3,4}^{(2)} - U_{1+2,1,2}^{(1)} U_{3+4,3,4}^{(1)} \\ &\times \left[\frac{1}{\omega_{1+2} - \omega_1 - \omega_2} + \frac{1}{\omega_{3+4} - \omega_3 - \omega_4} \right] \\ &- U_{-1-2,1,2}^{(3)} U_{-3-4,3,4}^{(3)} \\ &\times \left[\frac{1}{\omega_{1+2} + \omega_1 + \omega_2} + \frac{1}{\omega_{3+4} + \omega_3 + \omega_4} \right], \quad (A.1) \end{aligned}$$

$V_{1,2,3,4}^{(2)}$ is reported on page 16 in [20].

Below we report the analytical form of the coefficients $\mathcal{A}^{(i)}$ in (9):

$$\begin{aligned} \mathcal{A}_{0s,1s,2s}^{(1)} &= \frac{U_{0s,1s,2s}^{(1)}}{\omega_1 + \omega_2 - \omega_0}, \\ \mathcal{A}_{0s,1s,2s}^{(2)} &= 2 \frac{U_{2s,1s,0s}^{(1)}}{\omega_2 - \omega_1 - \omega_0}, \\ \mathcal{A}_{0s,1s,2s}^{(3)} &= -\frac{U_{2s,1s,0s}^{(3)}}{\omega_2 + \omega_1 + \omega_0}. \end{aligned} \quad (A.2)$$

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