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FLYING IN A SUPERFLUID: STARTING FLOW PAST AN AIRFOIL

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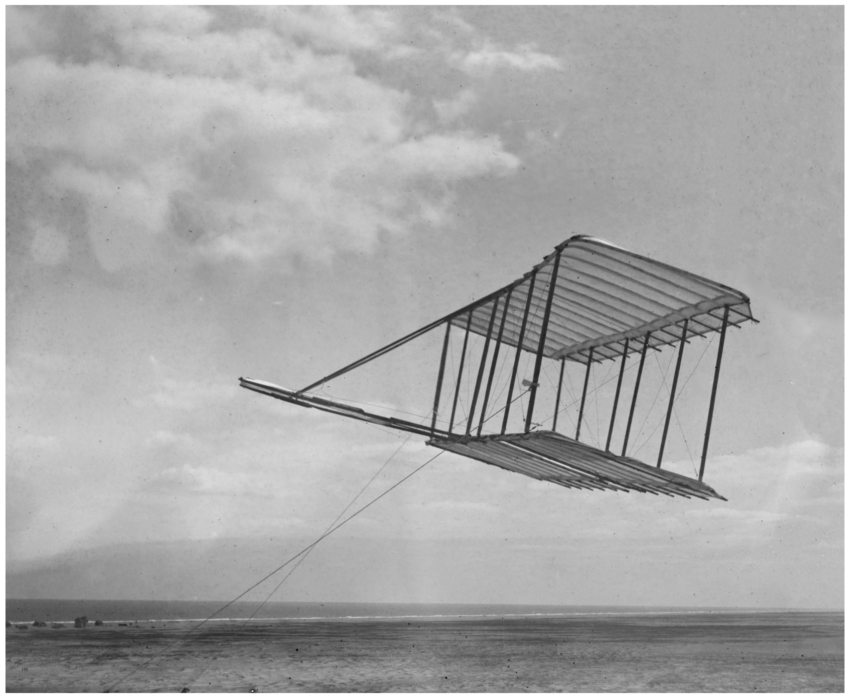
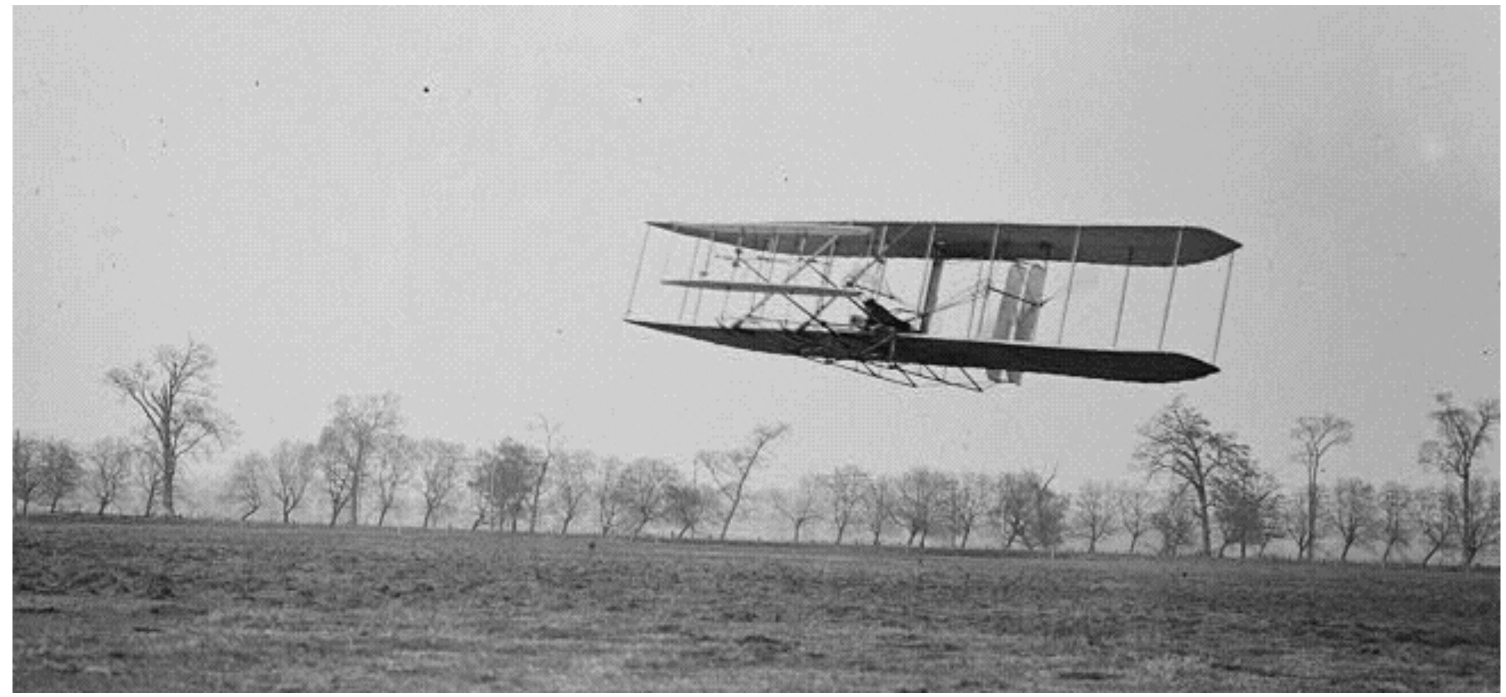
FLYING IN A SUPERFLUID: STARTING FLOW PAST AN AIRFOIL

- Recap on classical theory of flight: 2D and 3D
- Moving obstacles in superfluids
- How an airfoil potential may affect the superfluid flow

CLASSICAL THEORY OF FLIGHT



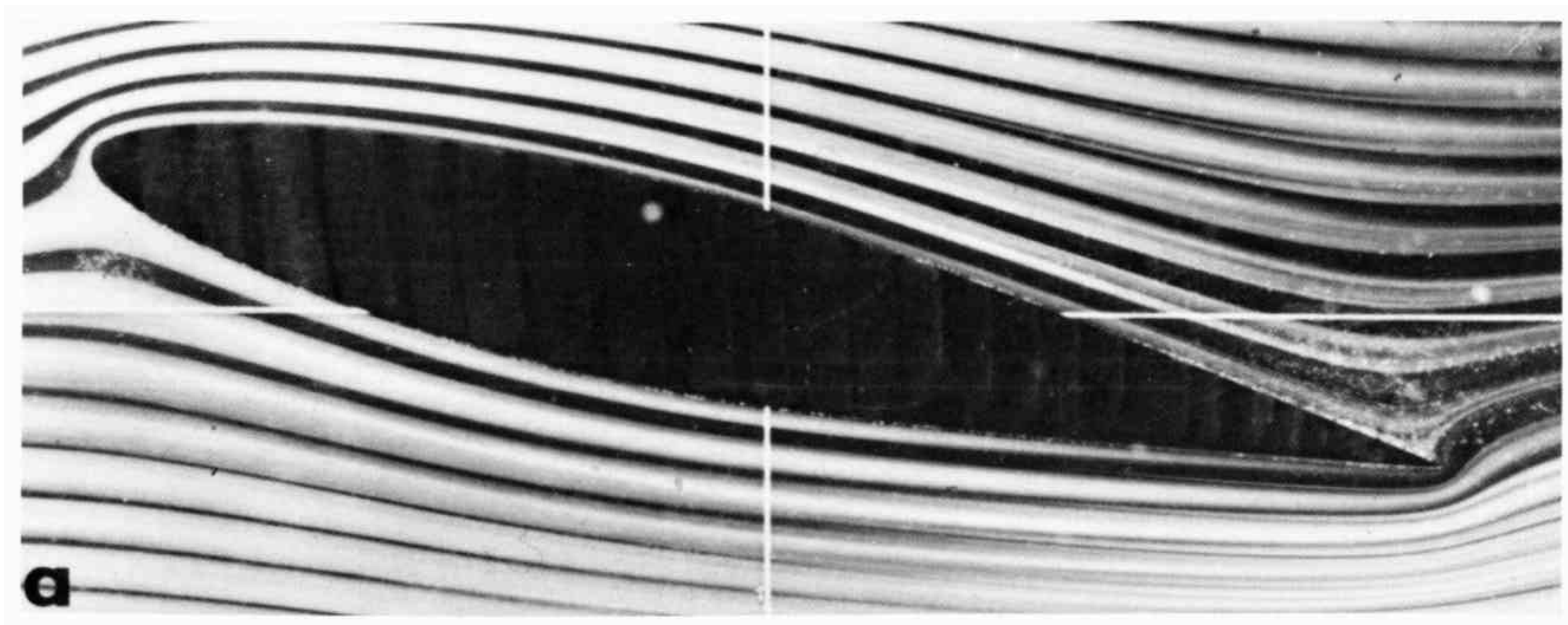


By Wright brothers - Library of Congress, Public Domain [Wikipedia]

- Inviscid theory to predict lift in stationary flow
- Viscous effects to explain the generation of lift and drag effects

[D.J. Achenson, Elementary Fluid Dynamics, Oxford University Press, 1990]

CLASSICAL THEORY OF FLIGHT



[M.Van Dyke, An Album of fluid Motion, 1982]

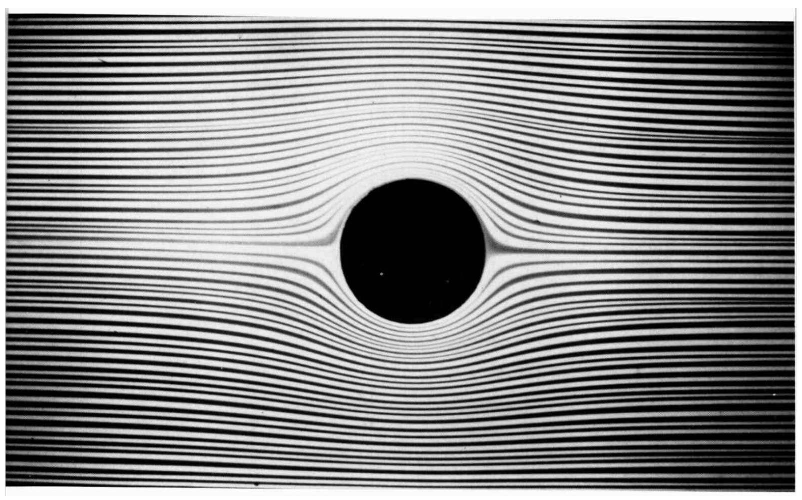
- The due to the positive angle of attack (or geometry) the fluid's speed is higher in the upper part of the airfoil (wing cross-section)
- The lift is a direct consequence of Bernoulli equation

$$\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\rho} = \text{const.}$$

2D INVISCID THEORY FOR AN AIRFOIL

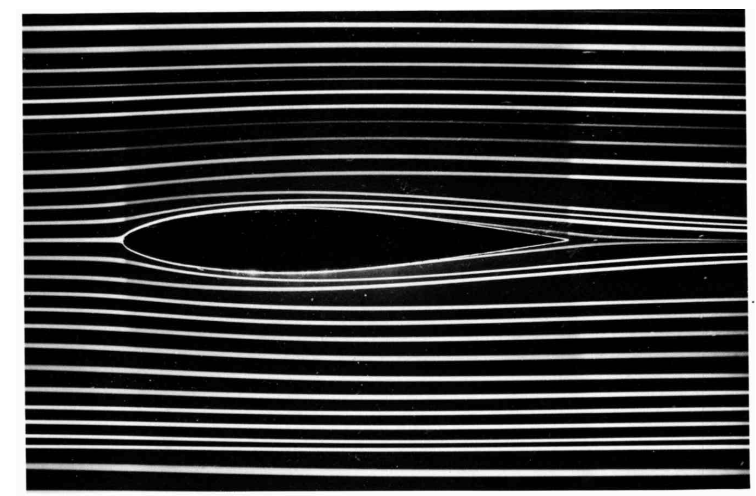
The two-dimensional flow resulting from the incompressible Euler equation past an airfoil can be analytically solved using conformal mapping

[M. Van Dyke, An Album of fluid Motion, 1982]



$$\rightarrow Z(z) \rightarrow$$

$$Z(z) = z + \frac{a^2}{z}$$

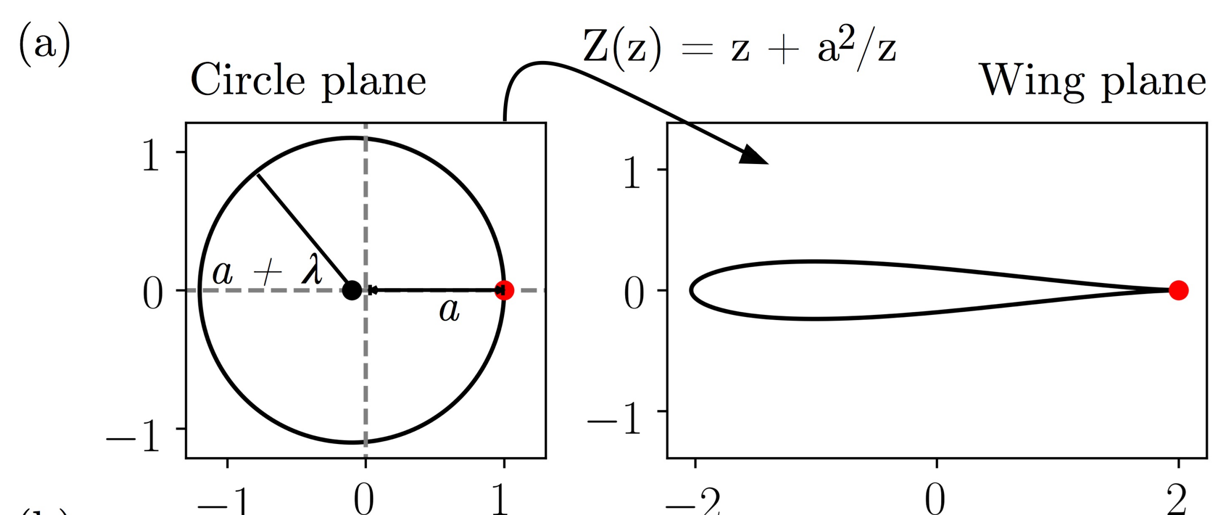


$$\frac{dw}{dz} = U_\infty \left(1 - \frac{a^2}{z^2}\right) - \frac{i\Gamma}{2\pi z}$$

- Complex velocity potential, solution of the flow past a cylinder

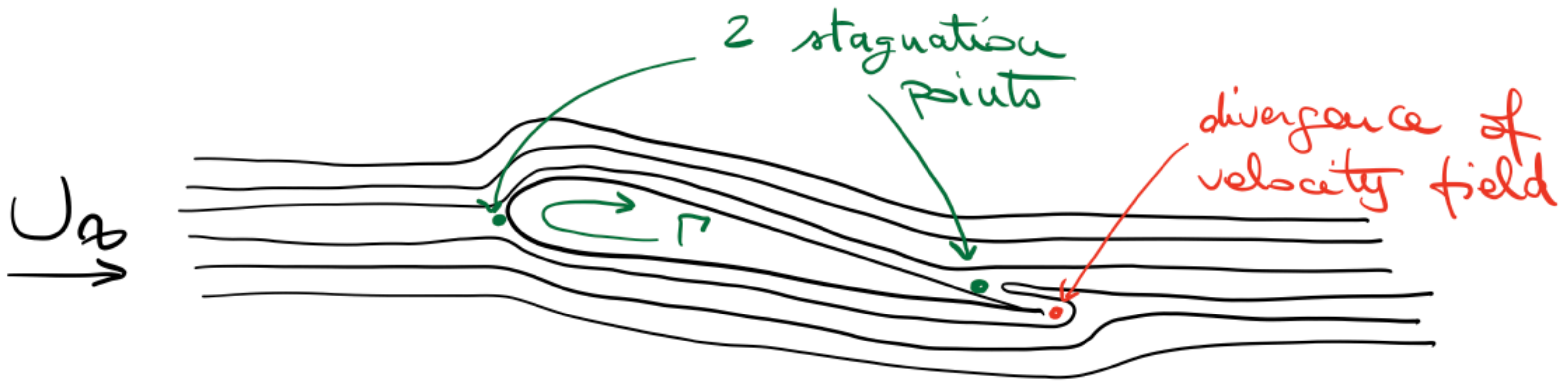
- Joukowski map, example mapping a circle onto an airfoil

  here $\lambda = -0.1, a = 1$



2D INVISCID THEORY FOR AN AIRFOIL

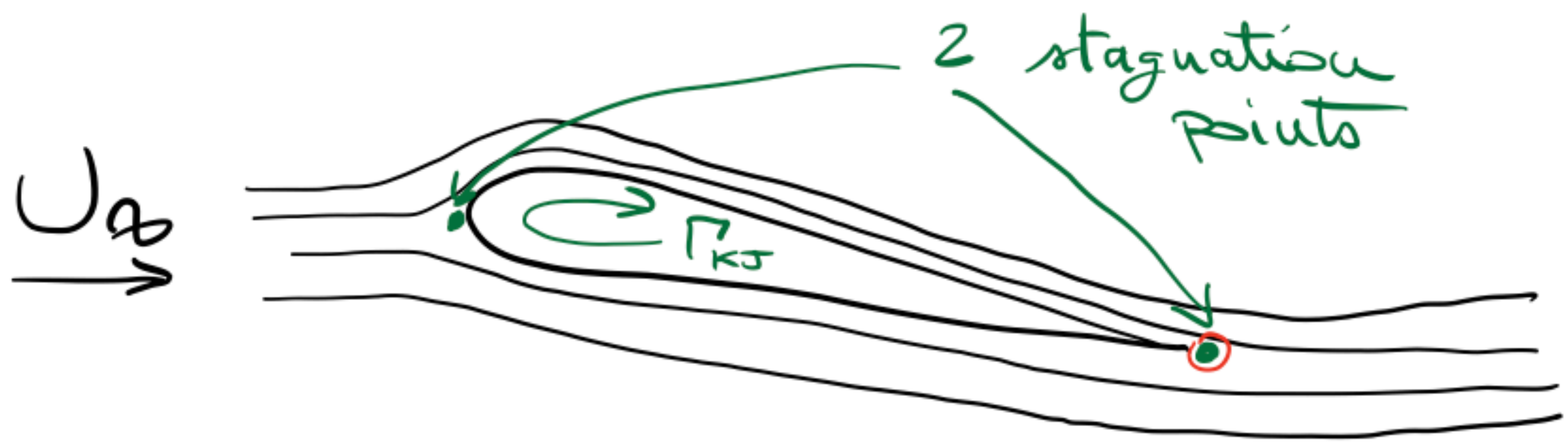
For a generic value of the terminal velocity, angle of attack, airfoil size and circulation around the airfoil, the streamlines in stationary conditions can be sketched as follows



- Two stagnation points (zero speed) at the airfoil, whose positions depend on the value of the circulation around the airfoil
- A divergence of the fluid's speed at the trailing edge of the airfoil due to the presence of a cusp

THE KUTTA-JOUKOWSKI CONDITION

For a generic value of the terminal velocity, angle of attack, airfoil size and circulation around the airfoil, the streamlines in stationary conditions can be sketched as follows



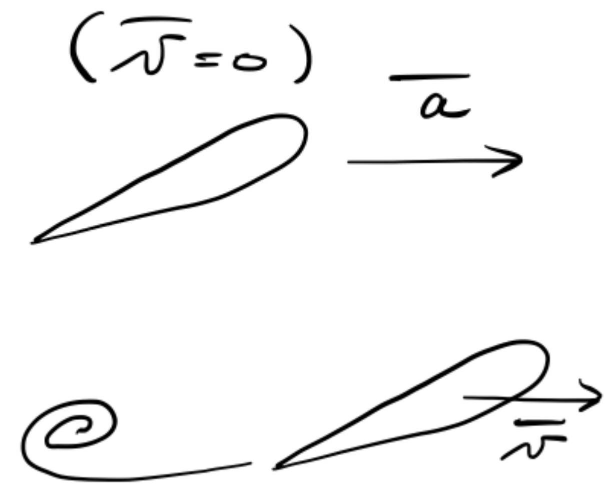
The unphysical divergence of the fluid speed is cancelled by letting one of the two stagnation points meeting the trailing edge. This mathematically results in the Kutta—Joukowski (KJ) condition

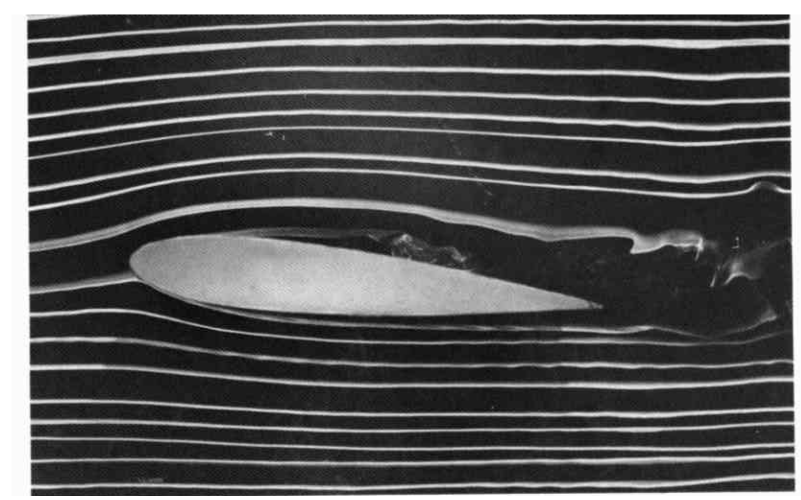
$$\Gamma_{KJ} = 4\pi U_\infty (a + \lambda)\ \sin\alpha$$

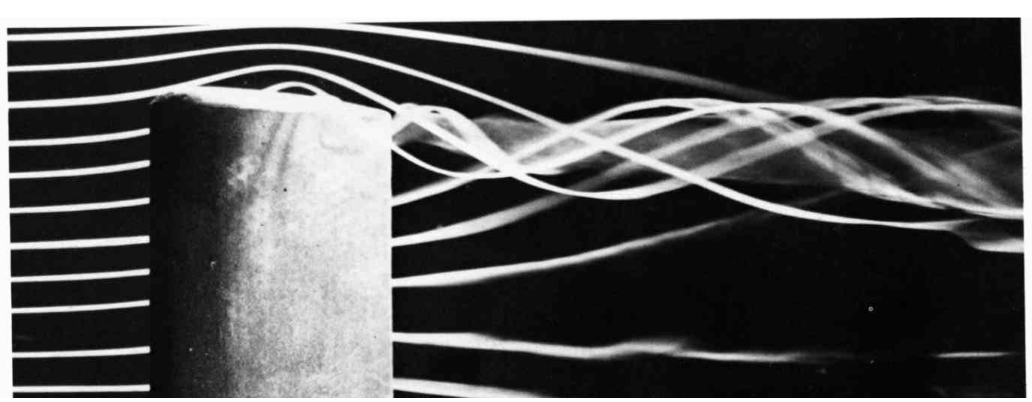
ADDING VISCOUS EFFECTS AND 3D CASE

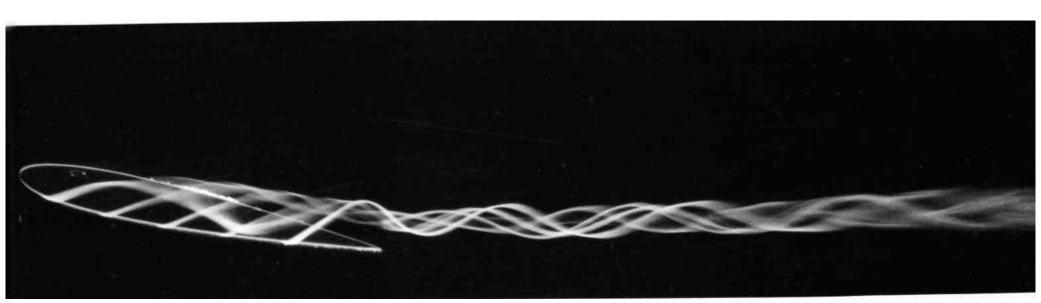
Viscous effects:

- cause generation of the KJ circulation around the airfoil (forbidden in inviscid fluid due to Helmoltz's third theorem)
- responsible for drag forces (form drag and skin drag)
- responsible for stall effect due to detachment of boundary layer









3D case:

- Vortex tubes created at the tips of the wings

Here not considered, only 2D!

FLYING IN A SUPERFLUID

- Can an accelerated airfoil acquire circulation?
- If so, what are the admissible values of the lift for a given airfoil, angle of attack and terminal velocity?
- Does the airfoil experience any drag?

THE GROSS-PITAEVSKII MODEL

$$\imath\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi - g\,|\psi|^2\psi - V_{ext}\psi = 0$$

- It is a mean-field equation that turns out to model incredibly well cold dilute Bose gases at very low temperature
- It also model qualitatively well superfluid liquid Helium
- In absence of the external potential, the ground-state is obtained for $|\psi_{GS}| = \sqrt{\rho_\infty}$
- The healing length $\xi = \sqrt{\hbar^2/(2mg\rho_0)}$ is the only inherent length scale of the system
- The large scale perturbation of the ground-state are phonon-like excitation of sound speed $c = \sqrt{g\rho_0/m}$

THE GROSS-PITAEVSKII MODEL

$$\imath\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi-g\,|\psi|^2\psi-V_{ext}\psi=0$$

Using Madelung transformation $\psi=\sqrt{\rho}\exp(\imath\phi)$ and defining density and velocity as $\rho=m\,|\psi|^2$ and $\mathbf{v}=\hbar/m\nabla\phi$, respectively, then

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0$$

$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=\nabla\left[-\frac{g}{m}\rho+\frac{1}{m}V+\frac{\hbar^2}{2m^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right]$$

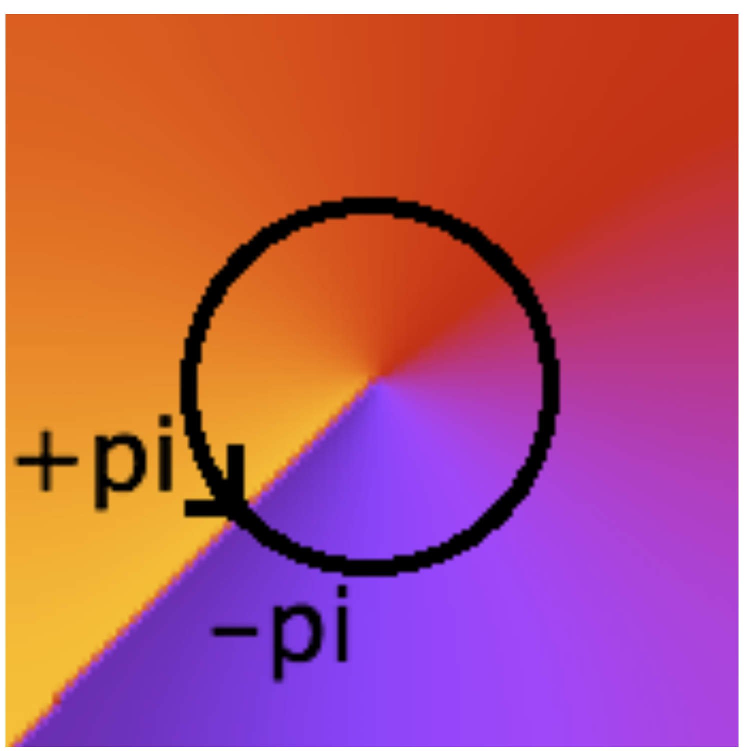
- The GP models an inviscid, barotropic, and irrotational fluid
- The last term of the second equation, the quantum pressure, becomes negligible at scales larger than the healing length ξ

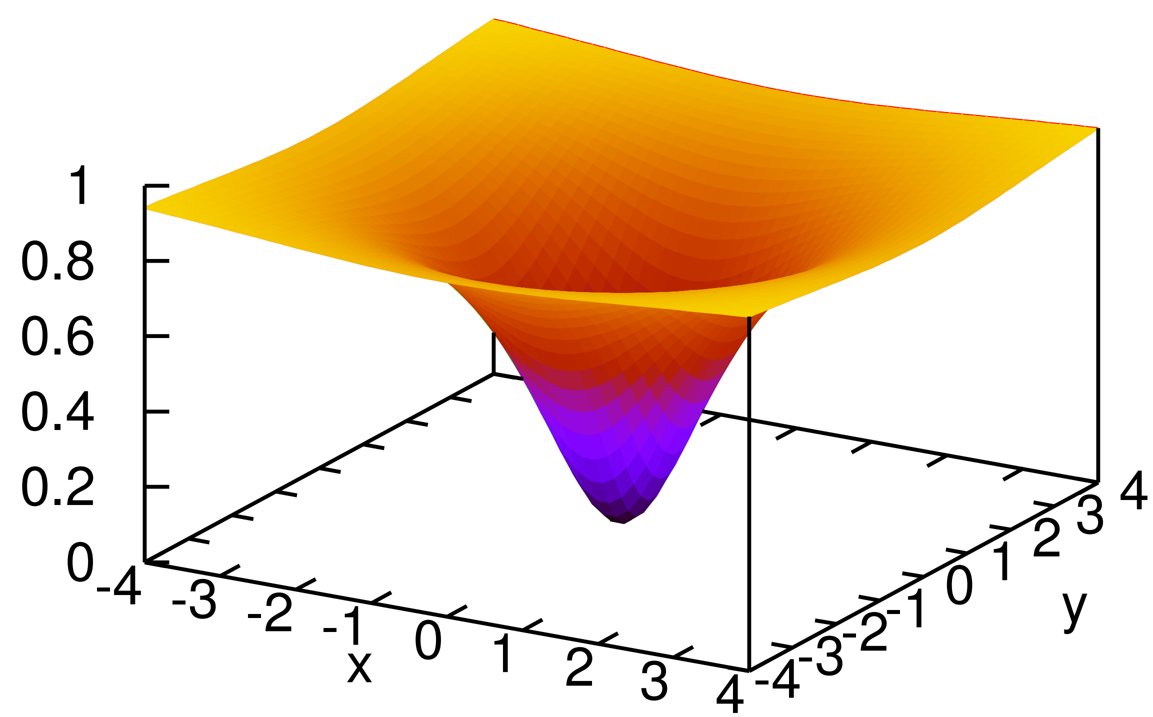
THE GROSS-PITAEVSKII MODEL

$$\imath\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi-g\,|\psi|^2\psi-V_{ext}\psi=0$$

Using Madelung transformation $\psi=\sqrt{\rho}\,\exp(\imath\phi)$ and defining density and velocity as $\rho=m\,|\psi|^2$ and $\mathbf{v}=\hbar/m\nabla\phi$, respectively, then

- Vortices are topological defect of the wave-function's argument

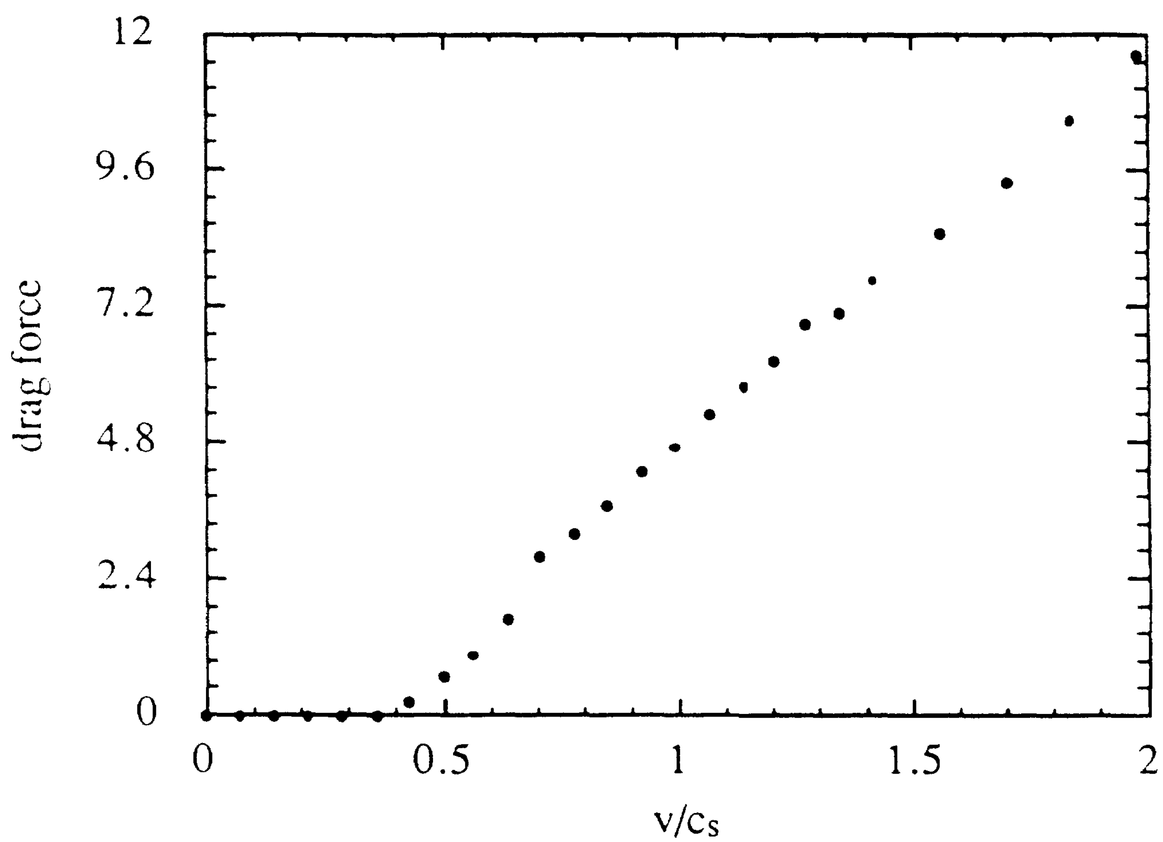


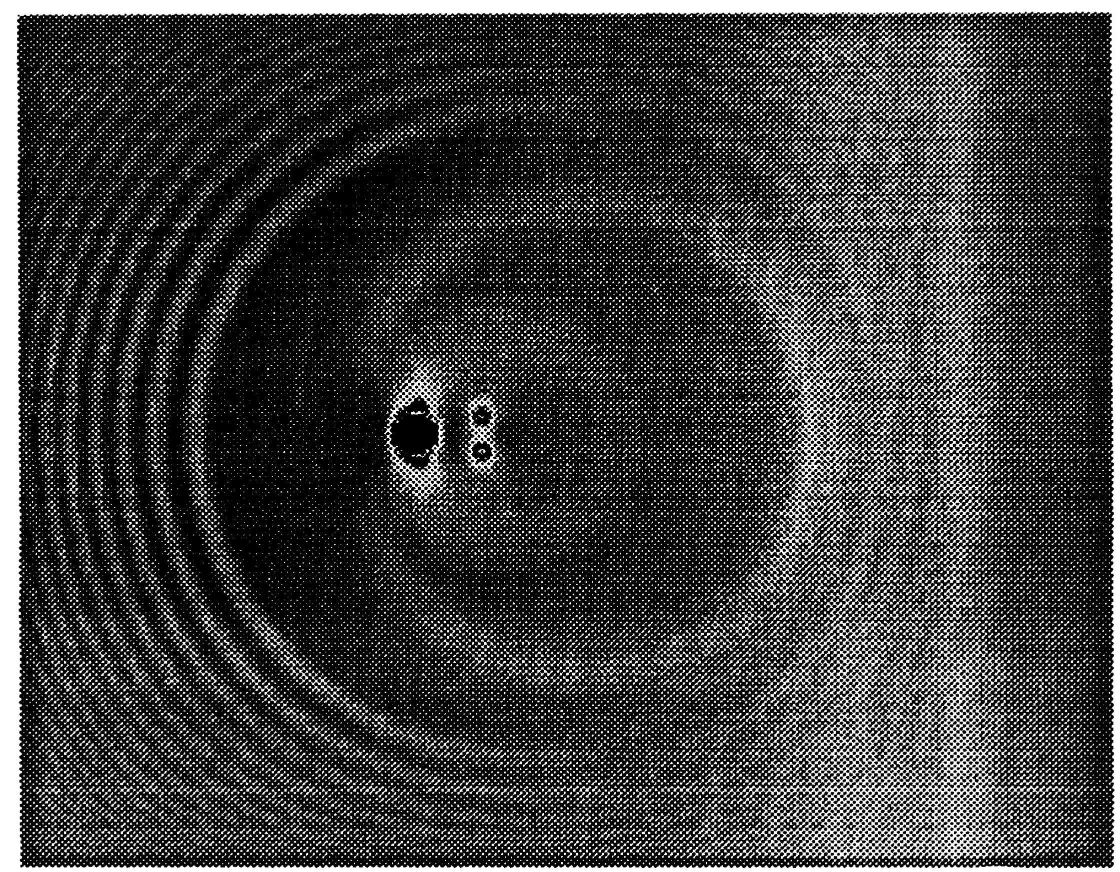


EXTERNAL POTENTIAL CYLINDER MOVING IN GP

An external potential moving in a superfluid may cause the flow to break the Landau's critical velocity (sound speed in GP), and generate excitations (travelling waves, solitons, vortices) and cause dissipation

<u>2d cylinder</u>



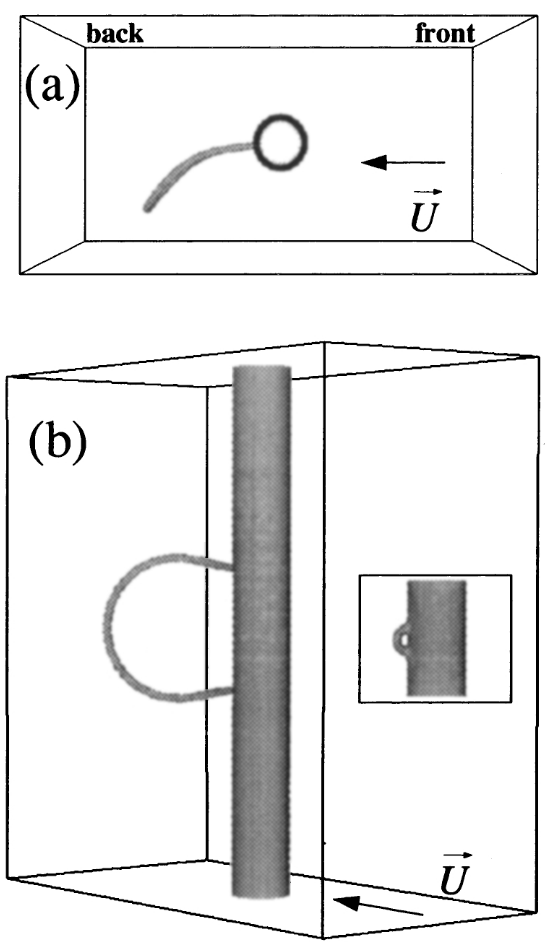


[Frisch et al., PRL 69, 1644 (1992)]

EXTERNAL POTENTIAL MOVING IN GP

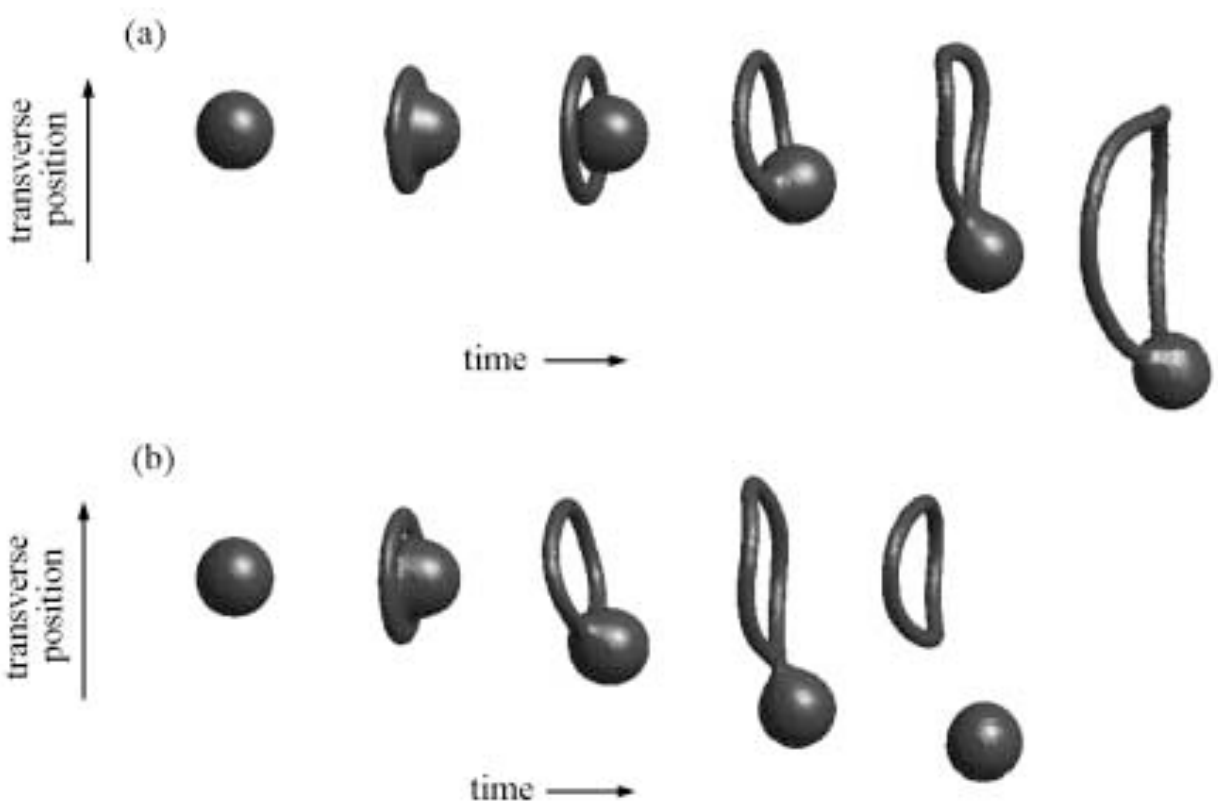
An external potential moving in a superfluid may cause the flow to break the Landau's critical velocity (sound speed in GP), and generate excitations (travelling waves, solitons, vortices) and cause dissipation

<u>3d cylinder</u>



[Nore et al., PRL 84, 2191 (2000)]

<u>3d sphere</u>

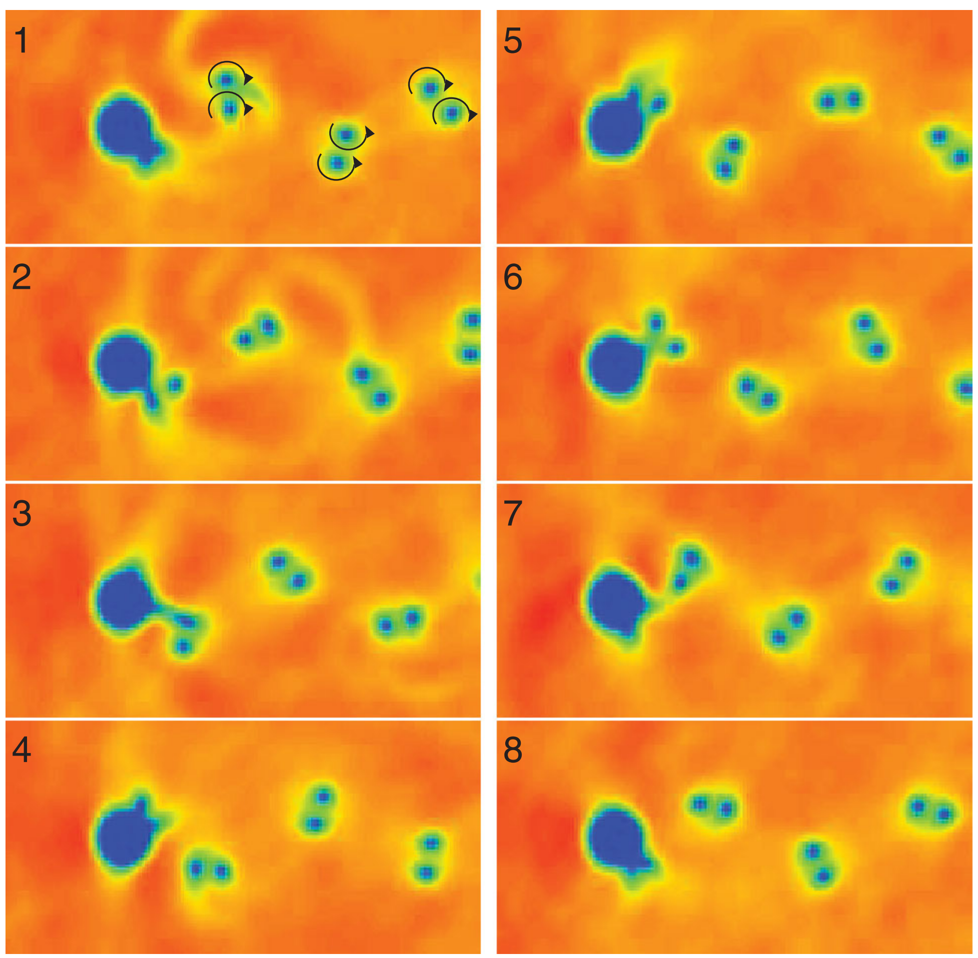


[Winiecki & Adams, Europhys. Lett. 52, 257-263 (2000)]

EXTERNAL POTENTIAL MOVING IN GP

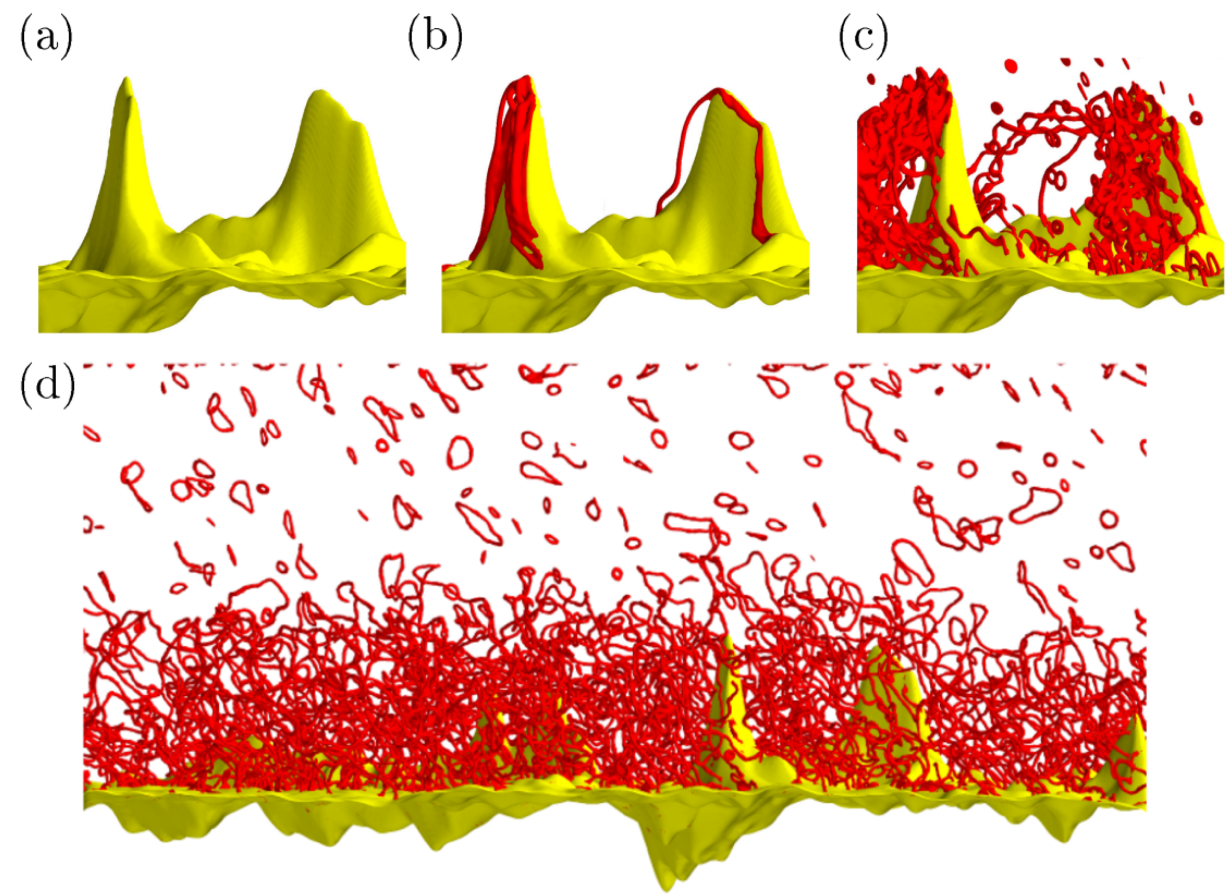
Some dynamical effects are very similar to the classical viscous ones

<u>Von Karman vortex sheet</u>



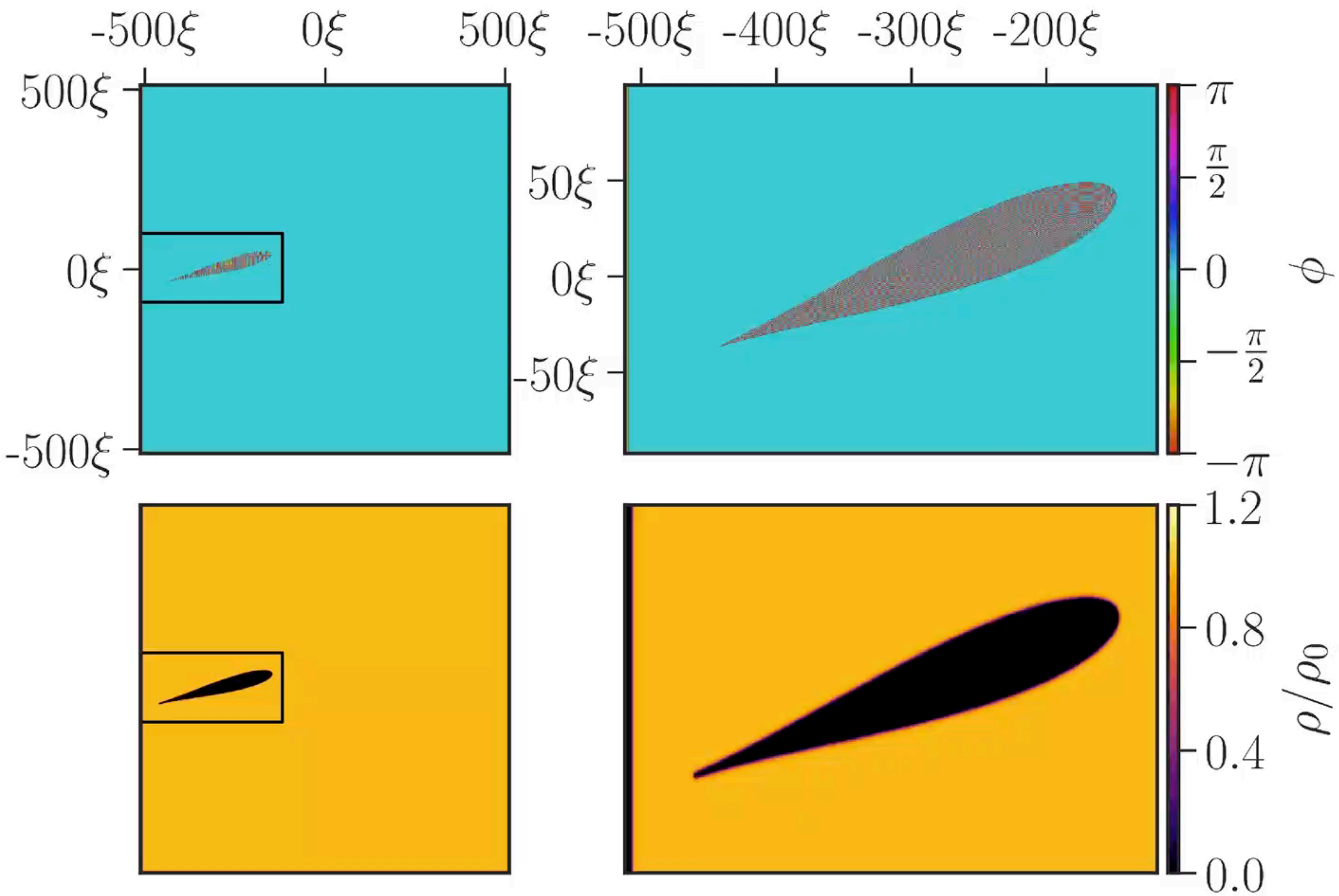
[Sasaki et al., PRL 104, 150404 (2010)]

<u>Boundary layer</u>



[Stagg et al., PRL 118, 135301 (2017)]

A TYPICAL SIMULATION



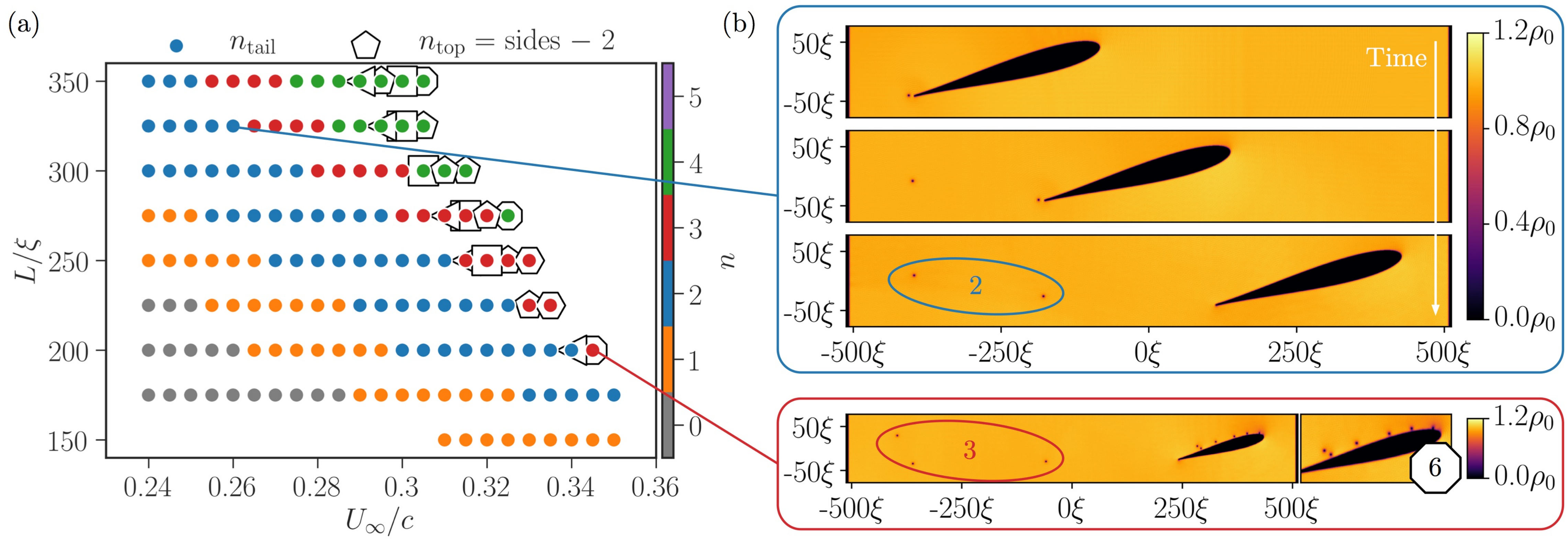
Top: evolution of the phase field.

Bottom: evolution of the superfluid density field.

- The airfoil moves initially with constant acceleration until it reaches a terminal velocity $U_\infty = 0.29c$
- The airfoil's length is $L = 325\xi$ and angle of attack $\alpha = \pi/12$
- Confining potential at the end of the computational box

EXPLORATION OF THE PARAMETERS SPACE

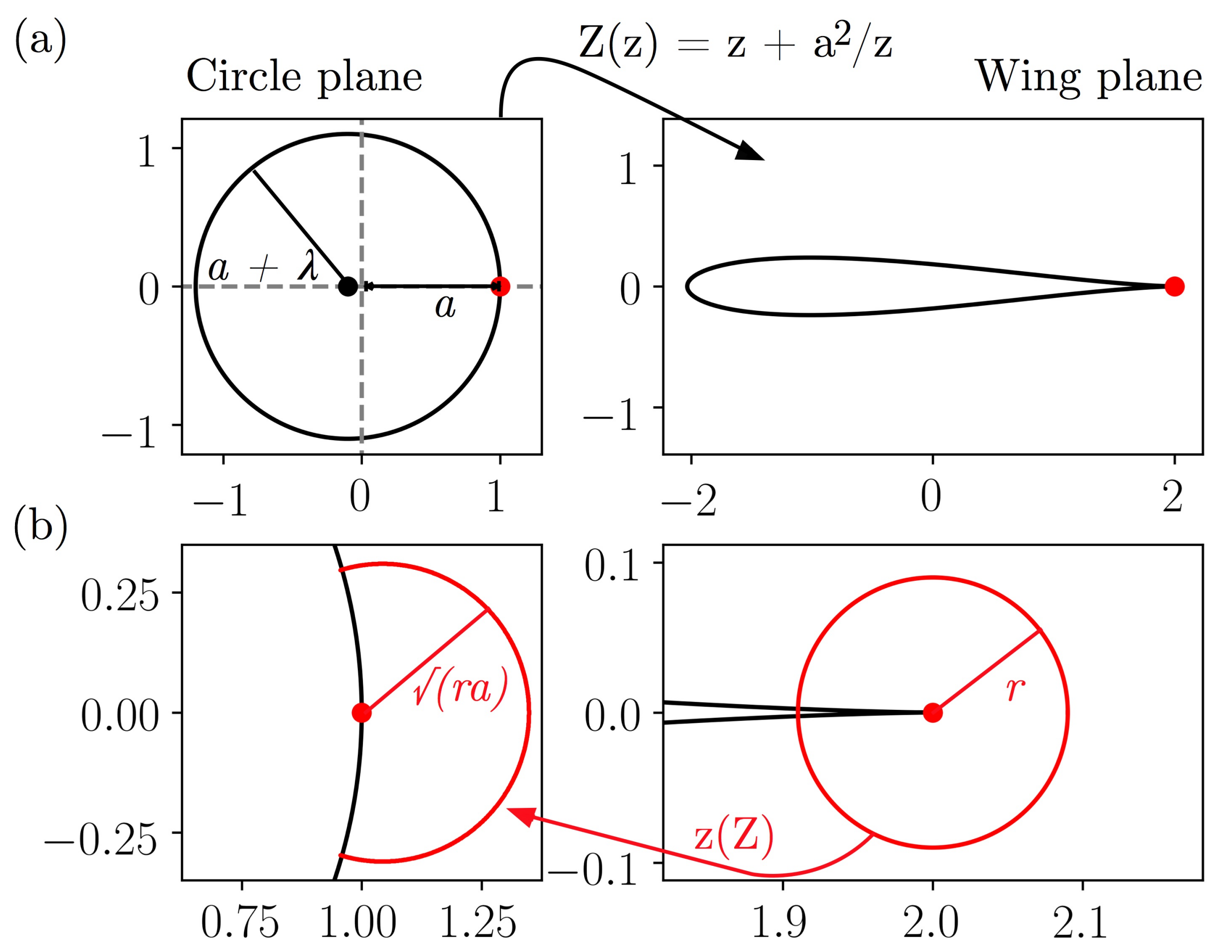
- We vary both the airfoil length and terminal velocity
- The airfoil shape ($\lambda = 0.1$) and angle of attack $\alpha = \pi/12$ are constant



Left: number of vortices produced at the trailing edge. Vortices produced at the top are highlighted with a polygon. Right: two simulation examples, the latter with the detachment of the boundary layer causing a stall condition.

HOW TO PREDICT THE NUMBER OF VORTICES GENERATED?

ASSUME INVISCID INCOMPRESSIBLE THEORY

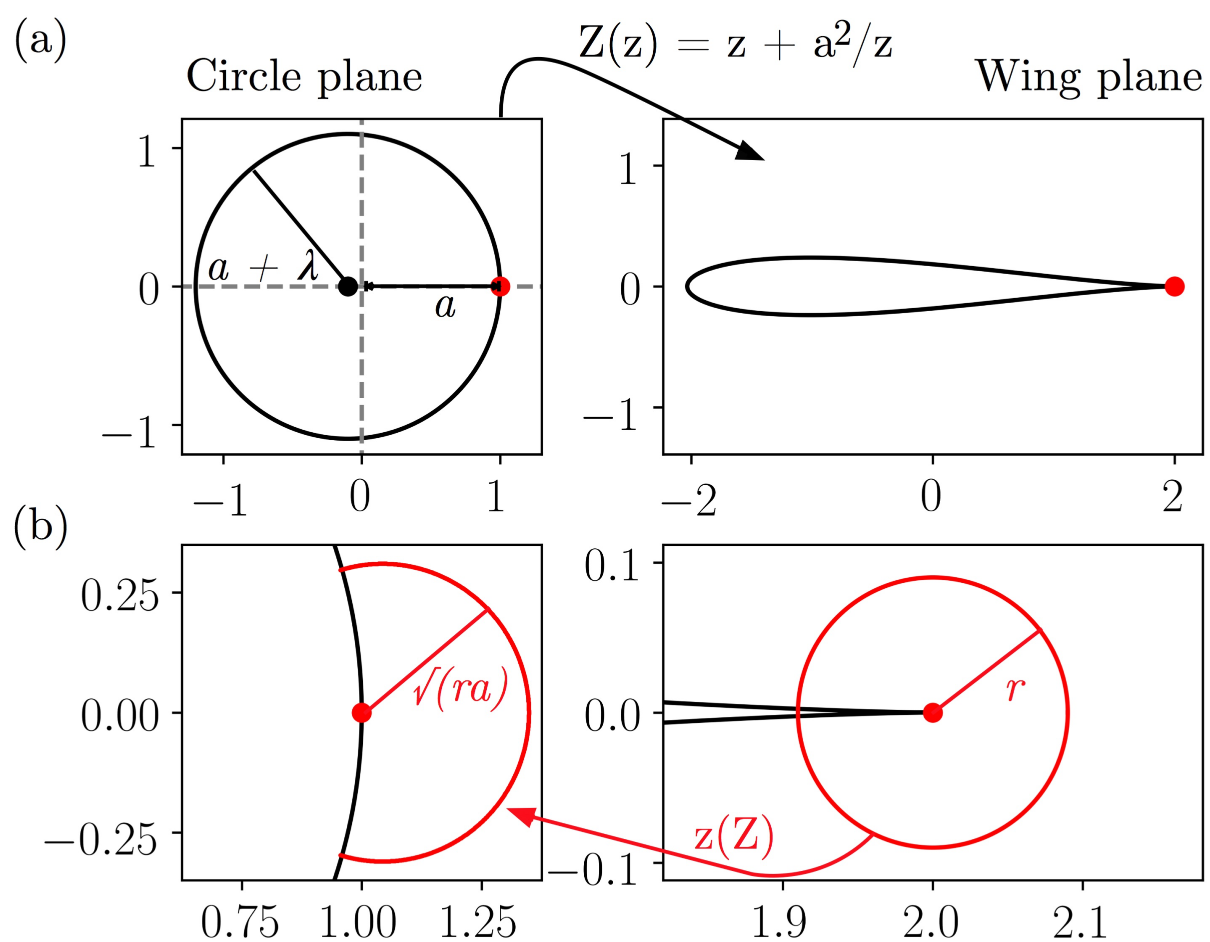


- Assuming steady flow
- Far from the healing layer around the airfoil assume incompressible inviscid theory (ideal theory) to hold

The magnitude of the velocity field around the trailing edge, Taylor-expanded about the Kutta—Joukowski condition results in

$$u_{\text{ideal}}^2 = \frac{1}{4}\frac{L}{r}U_\infty^2 \sin^2(\alpha)\left(1 - \frac{\Gamma}{\Gamma_{KJ}}\right)^2 + O\left(\sqrt{\frac{L}{r}}\right)$$

COMPRESSIBILITY CONDITION (NO QUANTUM PRESSURE)



- Assuming steady flow
- Far from the healing layer around the airfoil assume incompressible inviscid theory (ideal theory) to hold

The compressibility condition say that sound waves (and other excitations like vortices) occurs when the flow speed satisfies

$$\frac{3}{2}\frac{u_{\mathrm{ideal}}^2}{c^2} - \frac{1}{2}\frac{U_\infty^2}{c^2} - 1 > 0$$

IDEAL THEORY AND COMPRESSIBILITY CONDITION

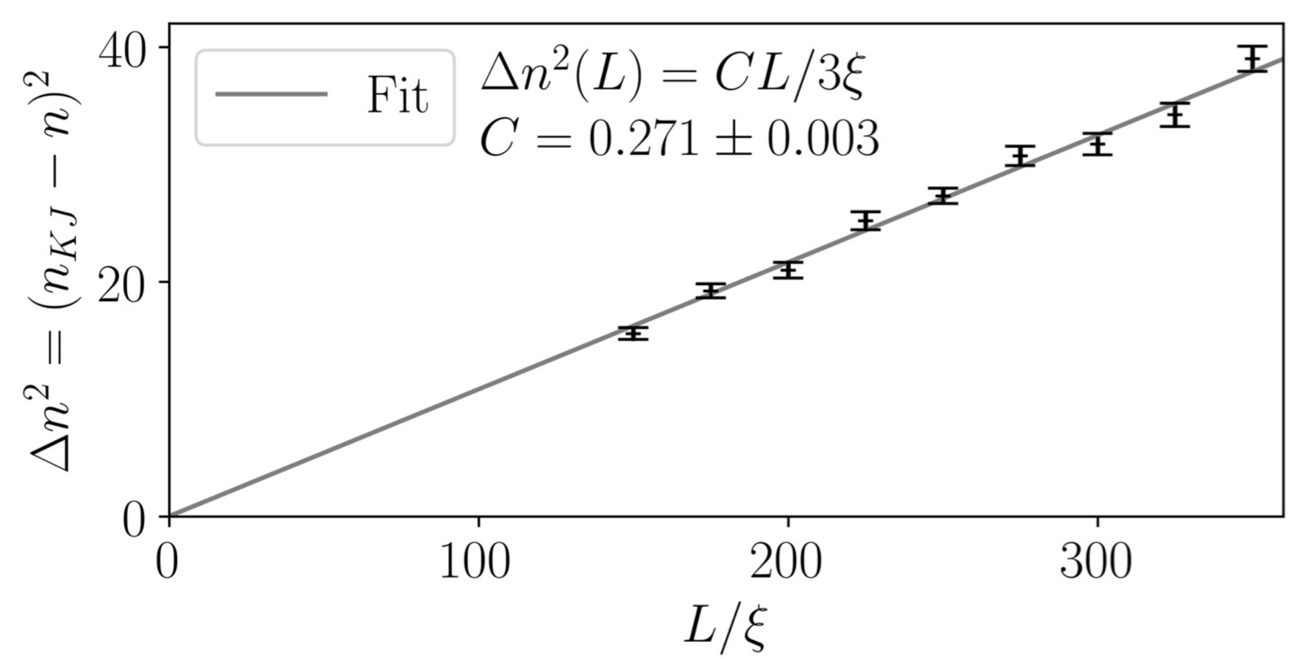
By assuming that the healing layer thickness is $r = \sqrt{C}\,\xi$ and combining the ideal theory predictions and the compressibility condition one finds that excitations are energetically favourable when

$$C \leq \frac{3}{8}\frac{L}{\xi}\left(\frac{U_\infty}{c}\right)^2 \sin^2(\alpha)\left(1 - \frac{\Gamma}{\Gamma_{KJ}}\right)^2$$

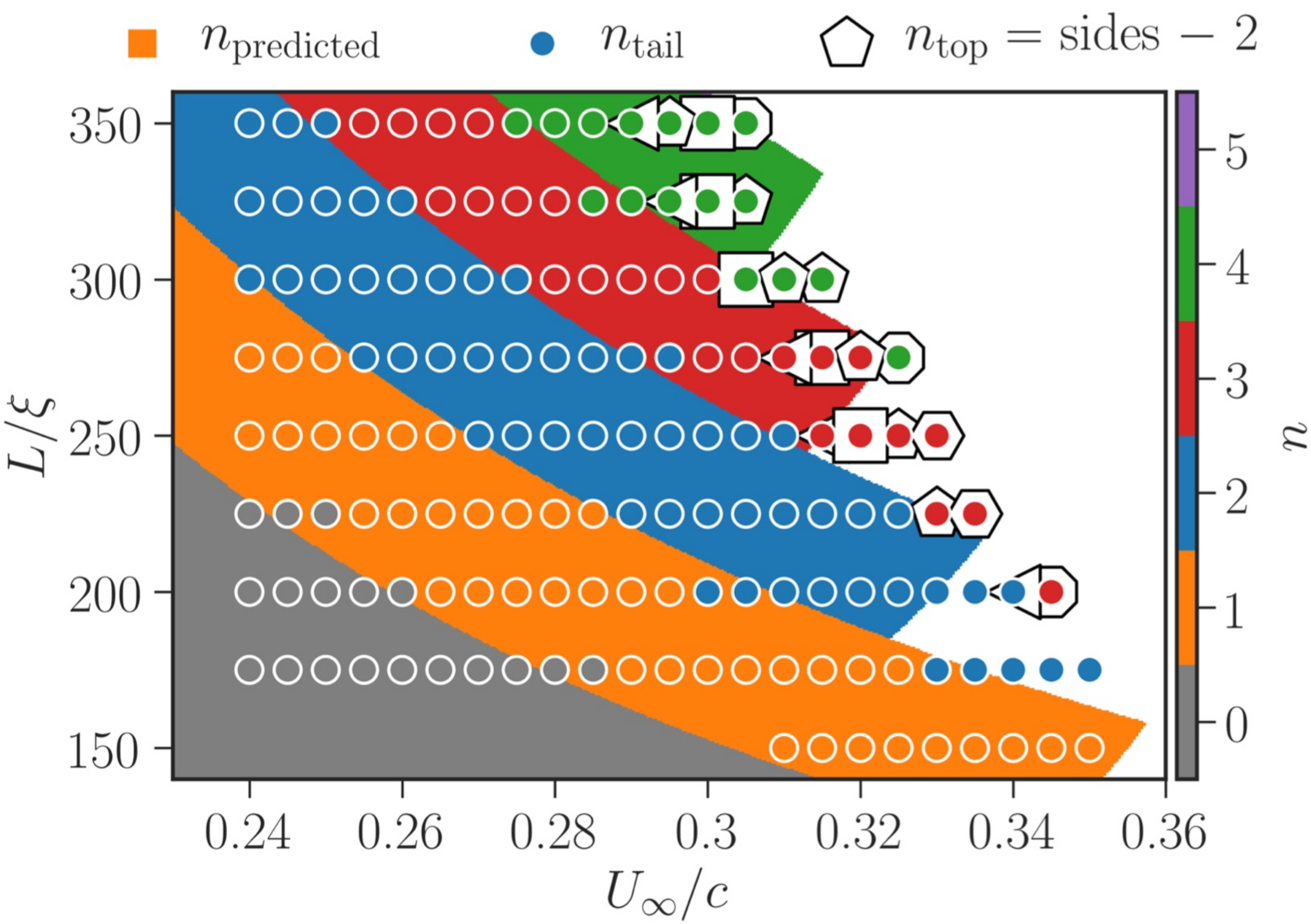
As in GP the circulation is quantised, $\Gamma = n\kappa$, with $n \in \mathbb{N}$, we can define by analogy $\Gamma_{KJ} = n_{KJ}\kappa$, with $n_{KJ} \in \mathbb{R}$

Rearranging the relation above and using the KJ condition one finds

$$\Delta n^2 = (n_{KJ} - n)^2 = CL/(3\xi)$$



PREDICTION OF THE NUMBER OF VORTICES

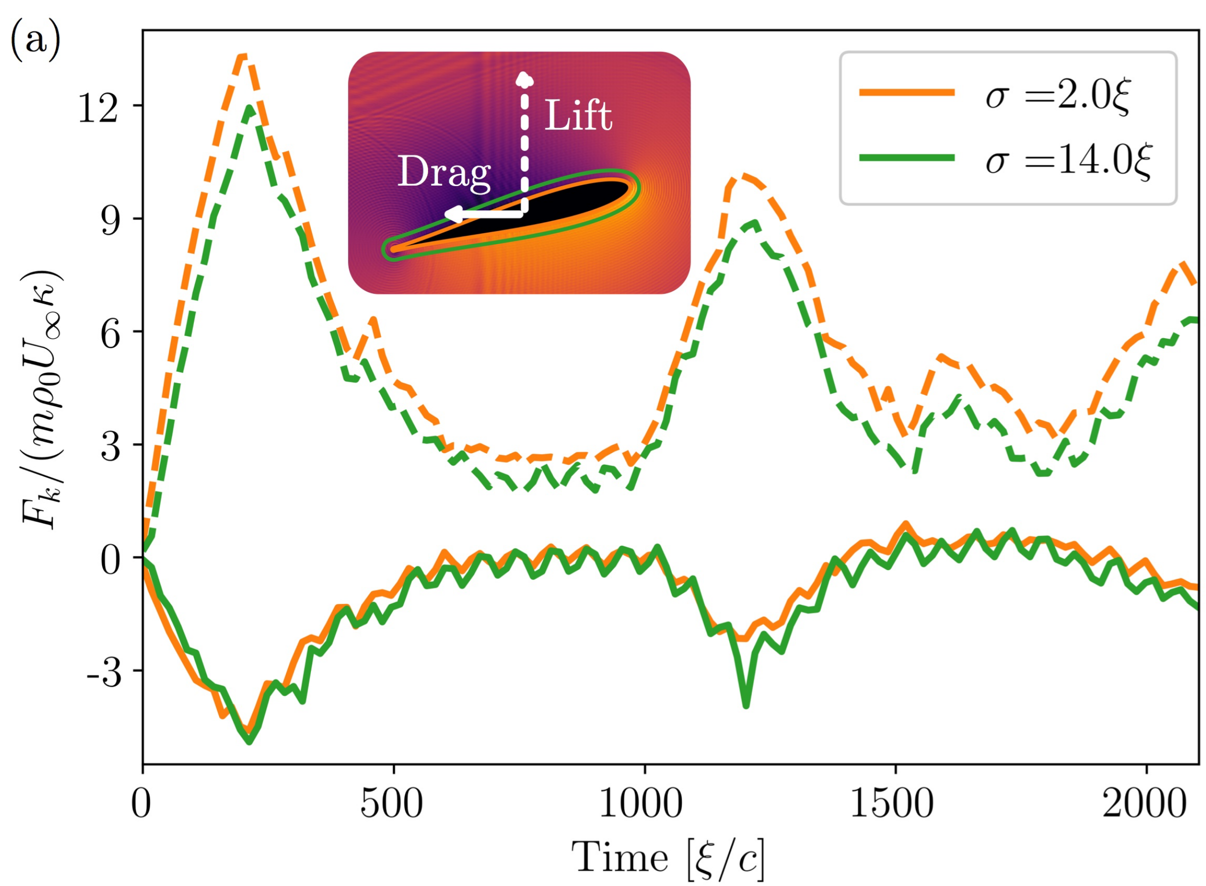


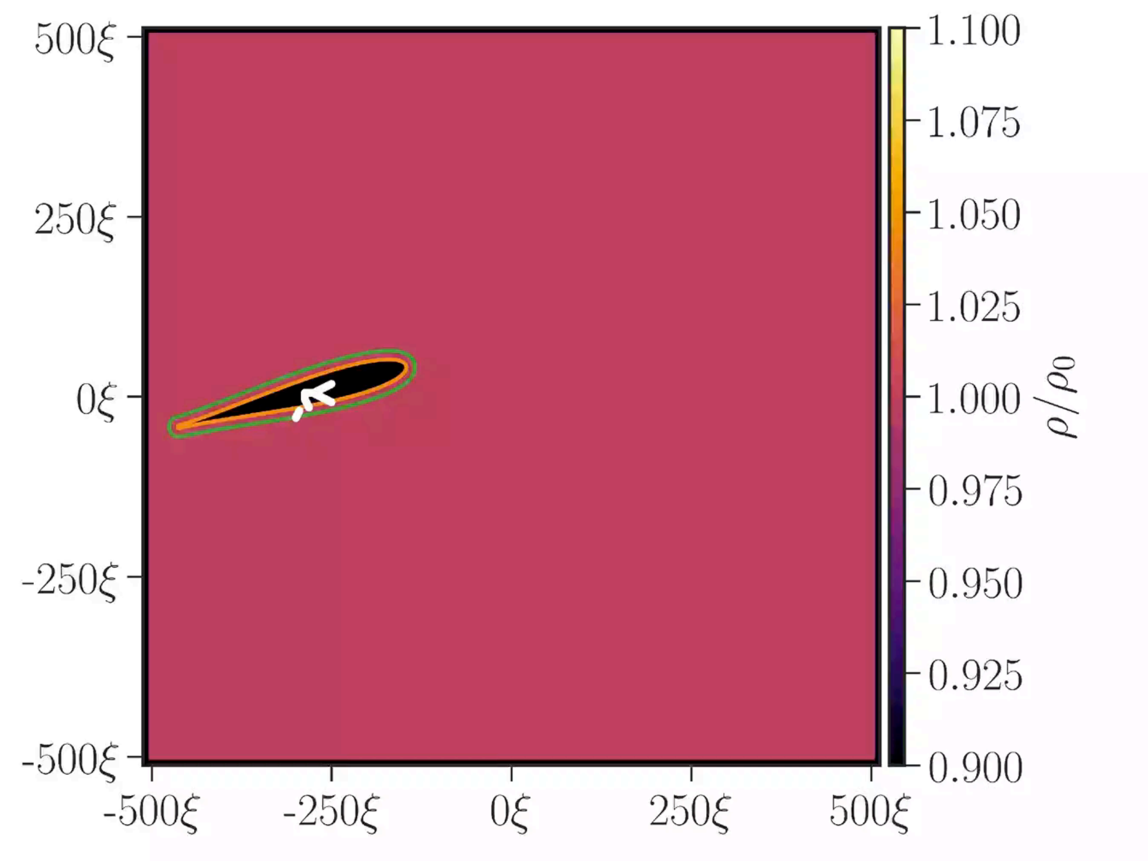
Number of vortices generated depending on the speed and length parameters. The curves indicate the phenomenological prediction. The white area indicate the stalling behaviour

Lift and drag can be measured integrating the stress-energy tensor

$$T_{jk} = m\rho U_j U_k + \frac{1}{2}\delta_{jk} g\rho^2 - \frac{\hbar^2}{4m}\rho\partial_j\partial_k \ln\rho$$

around a closed path containing the airfoil

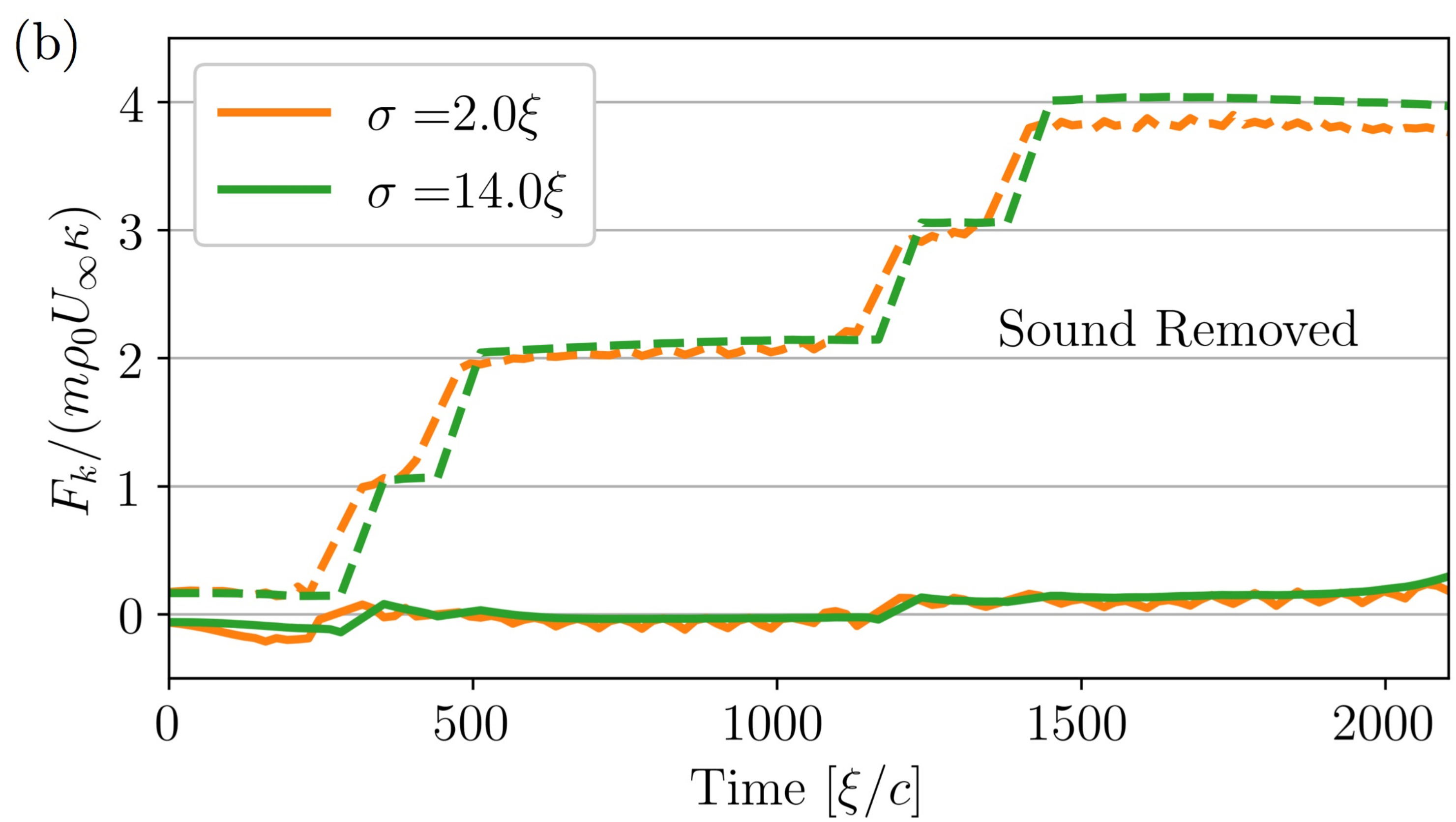




Left: rescaled lift (dashed) and drag (solid) versus time computed for different contours around the airfoil. Right: video showing the sound emission during the vortex nucleation at the trailing edge.

ABOUT LIFT AND DRAG (SOUND REMOVED)

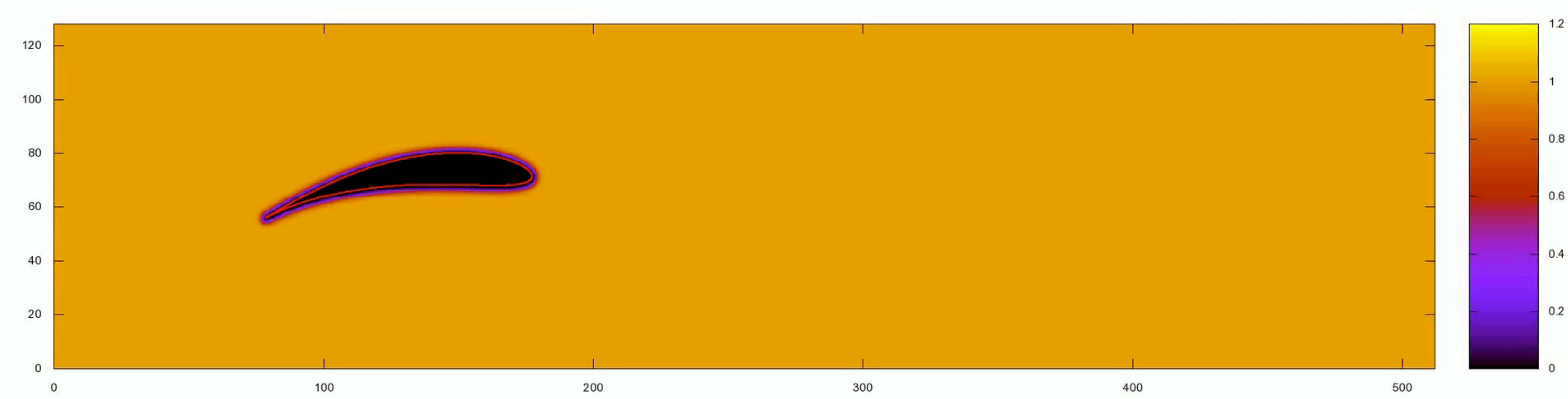
We can artificially remove the acoustic component in the velocity field by decomposing the velocity field into a compressible and incompressible component. The forces calculated with the acoustically-filtered velocity field and the density field prescribed by the Bernoulli equation become now



- Lift is quantised
- Drag becomes zero after the vortex nucleation

Left: rescaled lift (dashed) and drag (solid) versus time computed for different contours around the airfoil removing sound

CONCLUSIONS



- An airfoil moving in a superfluid can generate vortices at the trailing edge by breaking the Landau's critical speed
- To preserve the total circulation, the airfoil acquires a non-zero circulation
- This process is unsteady and generates sound
- When sound is removed (or steady regime is achieved) the airfoil experiences a quantised lift and no drag)
- If the terminal velocity of the airfoil is too high then a detachment of the boundary layer occurs (stall) and the steady regime cannot be achieved

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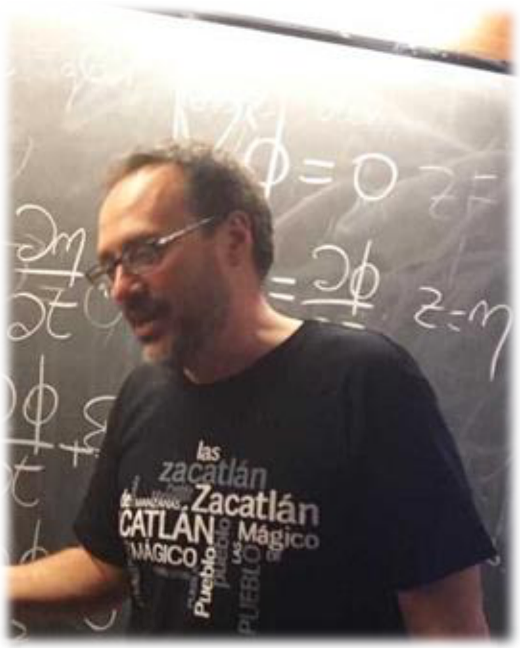
THANKS FOR YOUR ATTENTION!

Joint work with: Seth Musser, D.P., Miguel Onorato, William T.M. Irvine



Acknowledgments

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