DIRECT ENERGY CASCADE IN THE TWO-DIMENSIONAL GROSS-PITAESVKII MODEL

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OUTLINE OF THE TALK

- Brief introduction to the **Gross-Pitaevskii equation** that models a Bose-Einstein condensate, an example of a quantum fluid
- Discuss the main idea of the (weak) **wave turbulence theory** and its application to the Gross-Piteavskii model
- Simulations of out-of-equilibrium 3D and 2D systems, focussing in particular on the mechanisms carrying energy towards the small scales of the system, that is building a direct energy cascade
- **Bogoliubov turbulence**

THE GROSS-PITAEVSKII MODEL

$$i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi-g\,|\psi|^2\psi-V_{ext}\psi=0$$

Under a suitable dimensional rescaling and assuming for simplicity no external confinement one obtains the non-dimensional GP equation

$$i\partial_t\psi+\nabla^2\psi-|\psi|^2\psi=0$$

- Length scales are measured in units of the healing length $\xi=\sqrt{\dfrac{\hbar^2}{2mg\rho_\infty}}$
- It conserves particles (number of bosons) and energy, that is

$$N=\int|\psi|^2dV \quad \text{and} \quad H=\int|\nabla\psi|^2+\frac{1}{2}|\psi|^4dV$$

- Because it is an energy preserving dispersive nonlinear PDE (cubic nonlinear Schroedinger equation), it admits the **wave turbulence** (WT) theory approach

THE (WEAK) WAVE TURBULENCE THEORY APPROACH

The wave turbulence theory can be thought as a statistical mechanics approach to waves. It may be applied to any weakly nonlinear dispersive system like waves in optics, plasma, ocean, Bose-Einstein condensates, …

[Wave Turbulence, Nazarenko (2011)]

The efficient energy transfer in the system is mediated by only the resonant n-wave interaction processes satisfying

$$\begin{cases} \mathbf{k}_1 \pm \mathbf{k}_2 \pm \ldots \pm \mathbf{k}_n = 0 \\ \omega(\mathbf{k}_1) \pm \omega(\mathbf{k}_2) \pm \ldots \pm \omega(\mathbf{k}_n) = 0 \end{cases}$$

THE (WEAK) WAVE TURBULENCE THEORY APPROACH

In Fourier space, GP results in $\left(\tilde{\psi}_{\mathbf{k}} = \int \psi \exp\left[i\mathbf{k}\cdot\mathbf{x}\right] d\mathbf{x}\right)$

$$i\partial_t \tilde{\psi}_{\mathbf{k}_1} - \omega(\mathbf{k}_1)\,\tilde{\psi}_{\mathbf{k}_2} = \int \tilde{\psi}^*{}_{\mathbf{k}_2}\tilde{\psi}_{\mathbf{k}_3}\tilde{\psi}_{\mathbf{k}_4}\,\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,d\mathbf{k}_{234}\,, \quad \text{here} \quad \omega(\mathbf{k}) = |\mathbf{k}|^2$$

The WT introduces a statistical closure when **nonlinearity is weak**

- Statistical average over the highly fluctuating phase $\tilde{\psi}_i = |\tilde{\psi}_i|\,e^{\imath\theta_i}$
- Random phase approximation so higher order correlators are decomposed into lower ones (Wick decomposition)

$$\begin{cases} \langle\tilde{\psi}_i\rangle = \langle\,|\tilde{\psi}_i|\,e^{\imath\theta_i}\rangle = 0 \\ \langle\tilde{\psi}_i\hat{\psi}_j\rangle = \langle\,|\tilde{\psi}_i|\,|\tilde{\psi}_j|\,e^{\imath(\theta_i+\theta_j)}\rangle = 0 \\ \langle\tilde{\psi}_i\tilde{\psi}_j^*\rangle = \langle\,|\tilde{\psi}_i|\,|\tilde{\psi}_j^*|\,e^{\imath(\theta_i-\theta_j)}\rangle = n(\mathbf{k}_i)\,\delta(\mathbf{k}_i-\mathbf{k}_j) \\ \dots \\ \langle\tilde{\psi}_i\tilde{\psi}_j\tilde{\psi}_k^*\tilde{\psi}_l^*\rangle = n(\mathbf{k}_i)\,n(\mathbf{k}_j)\Big[\delta(\mathbf{k}_i-\mathbf{k}_k)\,\delta(\mathbf{k}_j-\mathbf{k}_l)+\delta(\mathbf{k}_i-\mathbf{k}_l)\,\delta(\mathbf{k}_j-\mathbf{k}_k)\Big] + C_{i,j,k,l} \end{cases}$$

- Predicts a **kinetic equation** to model the evolution of the spectrum $n(\mathbf{k}) \propto |\tilde{\psi}_{\mathbf{k}}|^2$
- WT theory for 2d/3d BECs

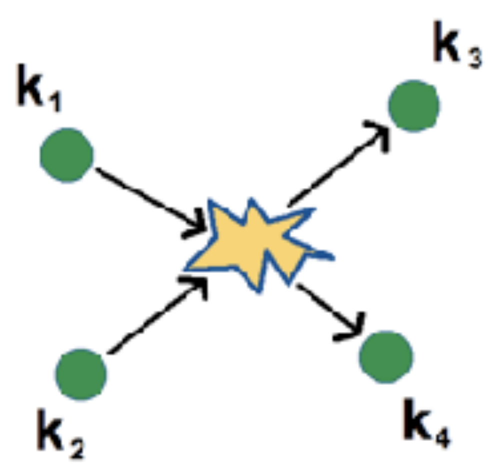
[Nazarenko & Onorato, Physica D 219, 1 (2006); Numasato et al., PRA 81, 063630 (2010); Nowak et al., PRA 85, 043627 (2012); Fujimoto & Tsubota, PRA 91, 053620 (2015)]

DE BROGLIE LIMIT, 4-WAVE KINETIC EQUATION

De Broglie limit is the limit where no modes are macroscopically occupied (no strong condensate), 4-wave kinetic equation

$$\frac{\partial n_1}{\partial t} = 4\pi \int n_1 n_2 n_3 n_4 \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right) \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$$

$$\times\, \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)\, d\mathbf{k}_{234}, \quad \omega_i = |\,\mathbf{k}_i|^2$$



Schematic of resonant 4-wave interactions

- Only resonant interactions contribute

$$\begin{cases} \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 \\ \omega_1 + \omega_2 = \omega_3 + \omega_4 \end{cases}$$

- Equilibrium is the **Rayleigh-Jeans distribution**

$$n_{RJ}(\mathbf{k}) = \frac{T}{\mu + \omega(\mathbf{k})}$$

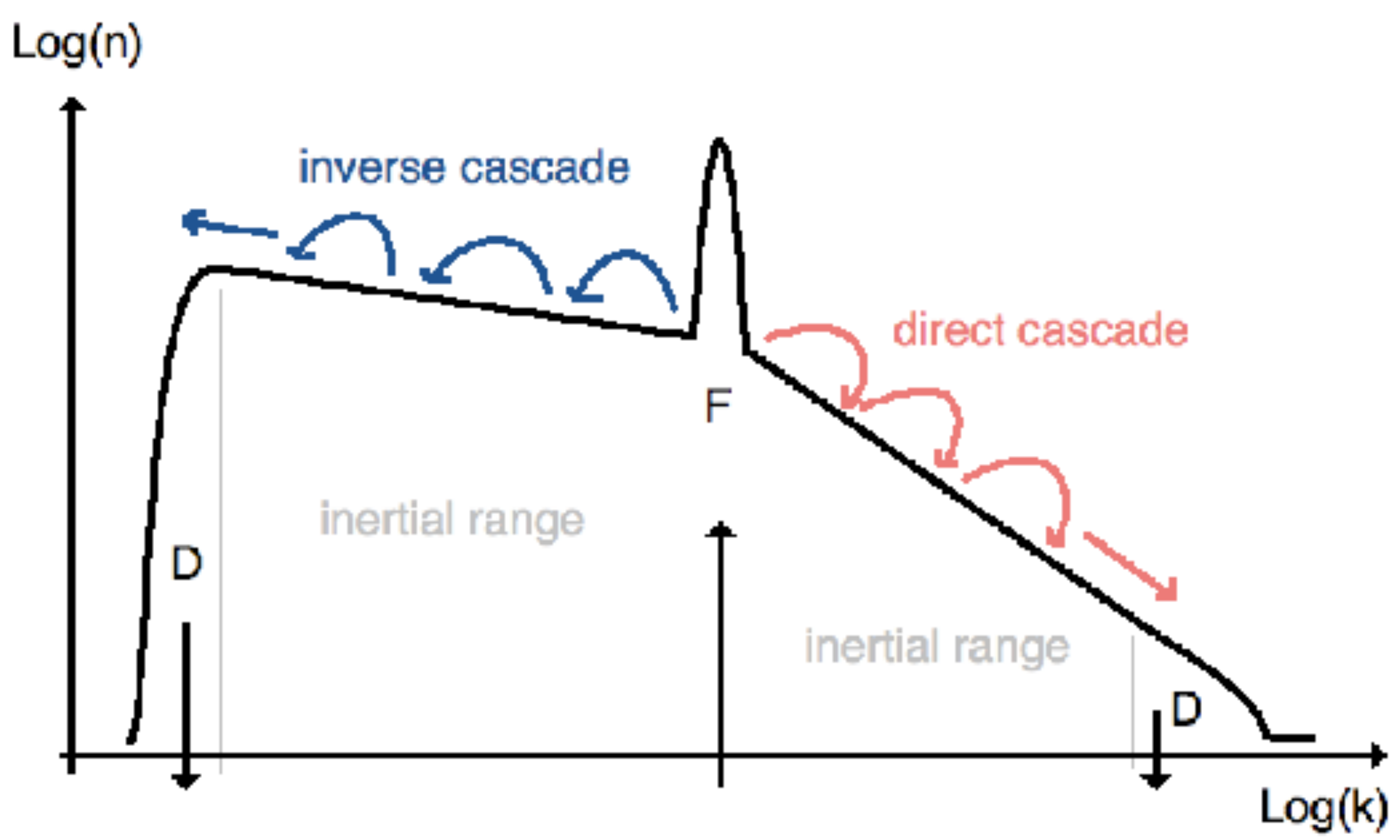
- Existence of other steady state distributions in the form of power-laws, called **Kolmogorov-Zakharov solutions**, carrying a constant flux of conserved quantities through scales

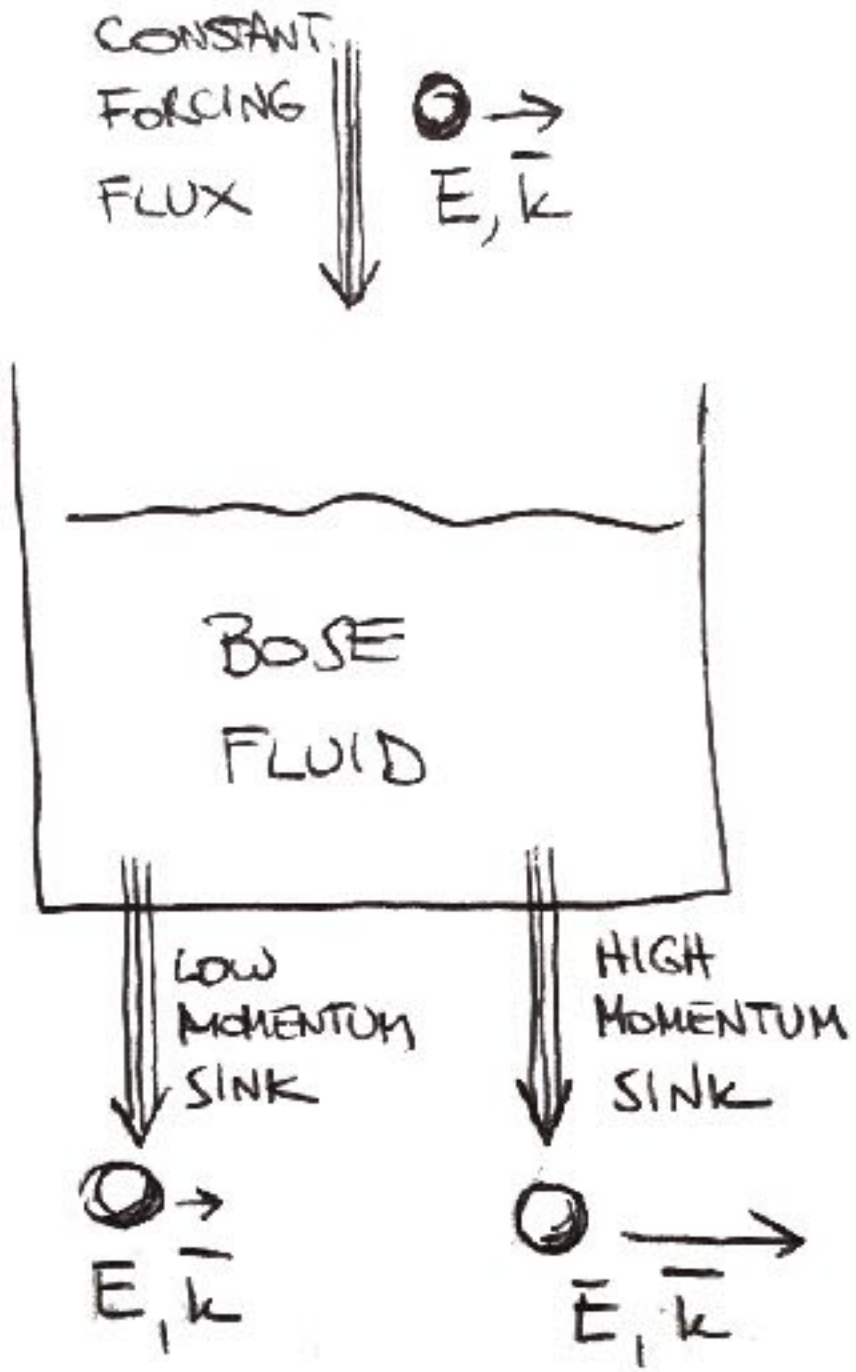
KOLMOGOROV-ZAKHAROV CASCADE SOLUTIONS (IN 3D)

Assuming the system is isotropic

$$n_{1D}(k) = \int n(\mathbf{k})\, d\Omega \propto n(\mathbf{k})\, k^{d-1} \quad \text{given} \quad k = |\mathbf{k}|$$

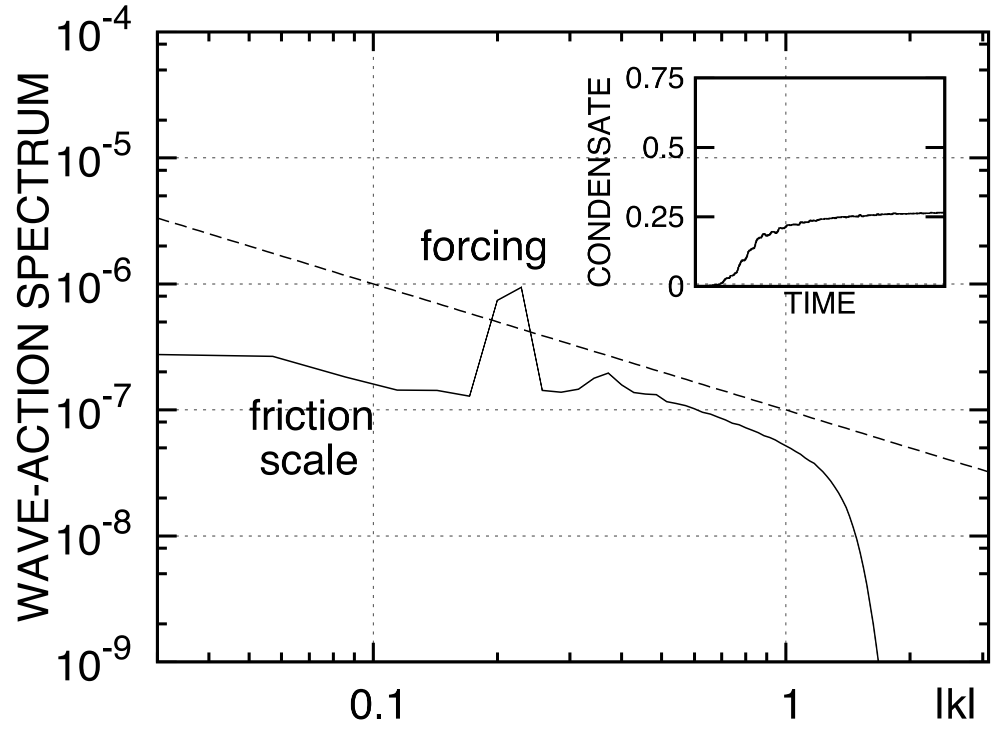
- Direct energy cascade $n_{1D}(k) \propto k^{-1}$
- Inverse particles cascade $n_{1D}(k) \propto k^{-1/3}$





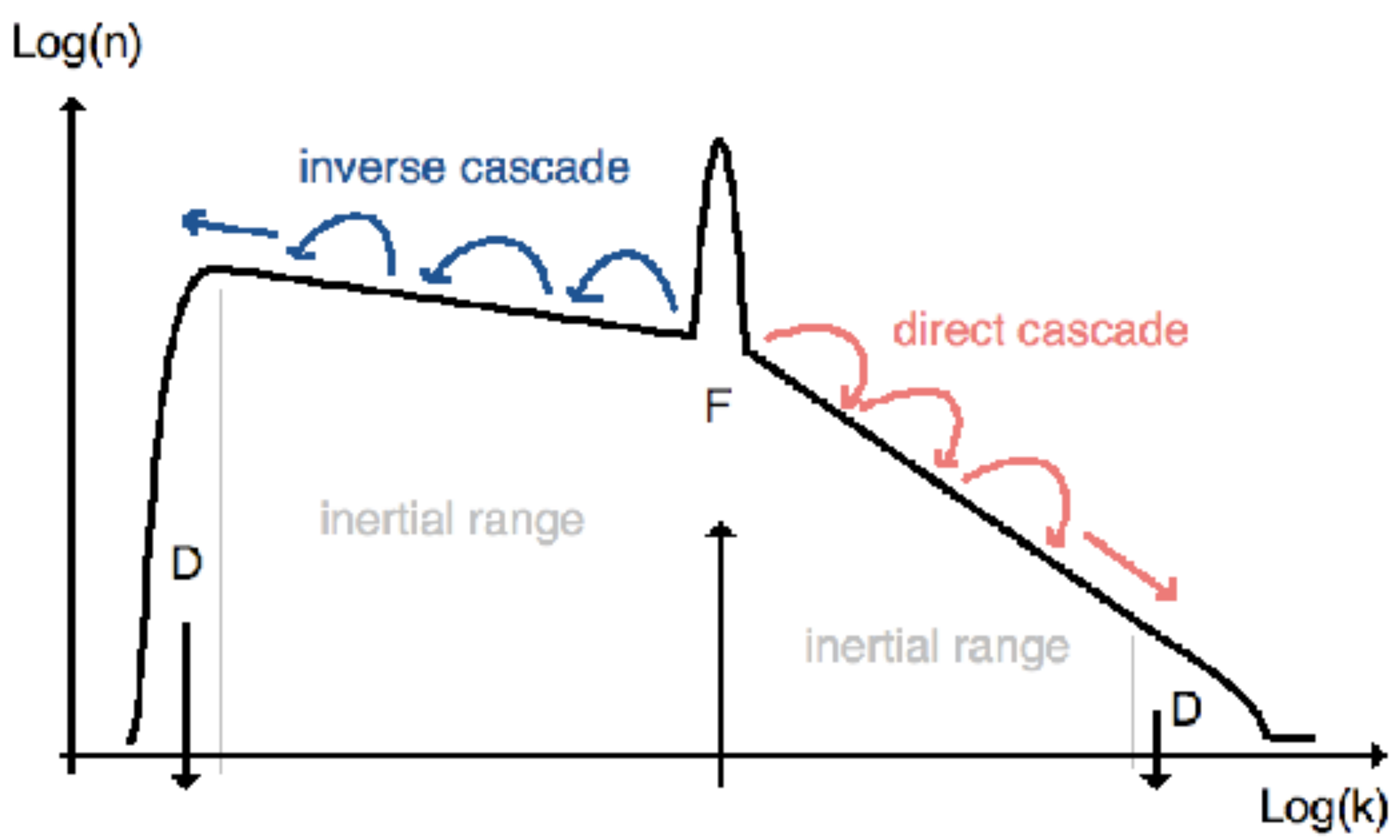
Sketch of the process

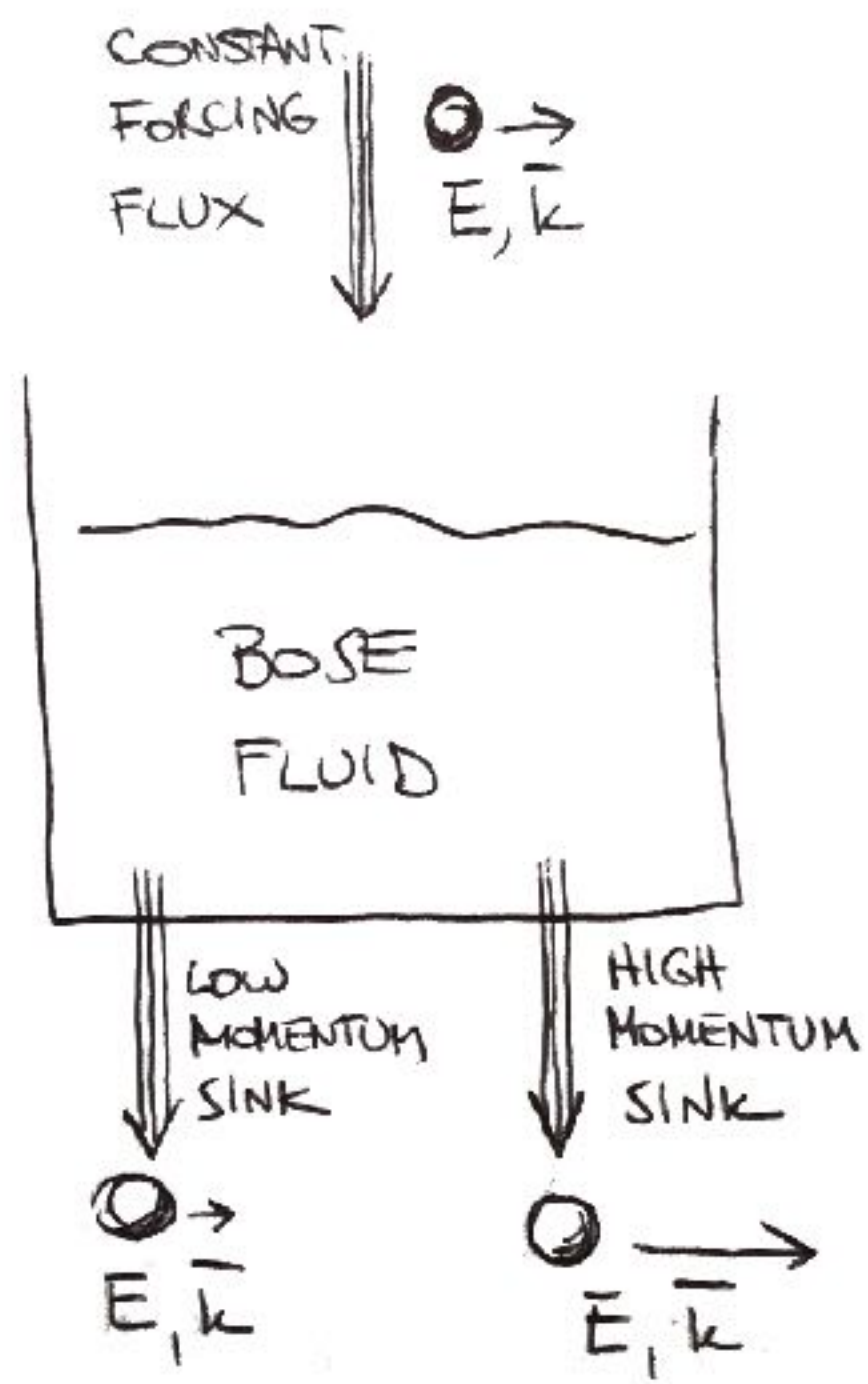
KOLMOGOROV-ZAKHAROV CASCADE SOLUTIONS (IN 3D)



$$n_{1D}(k) \propto k^{-1}$$

[D. P., S. Nazarenko, and M. Onorato, PRA 80, 051603(R) (2009)]

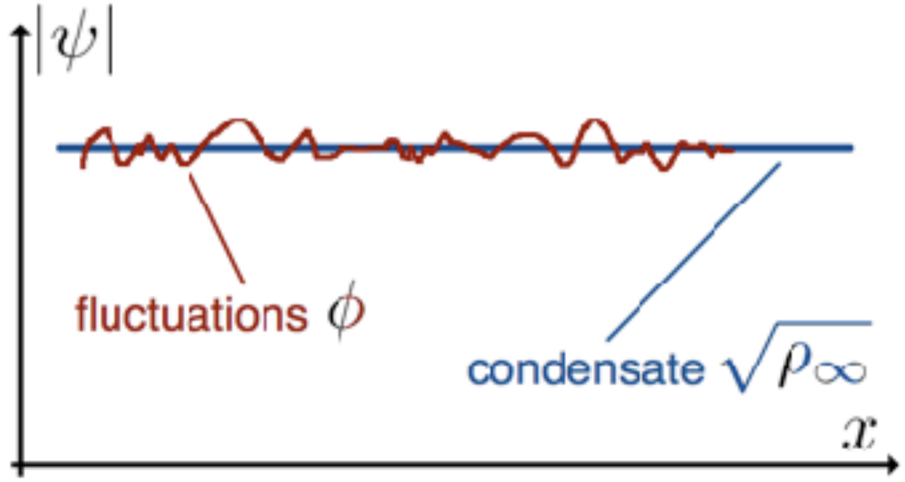




Sketch of the process

BOGOLIUBOV LIMIT, 3-WAVE KINETIC EQUATION

The WT theory can be also applied in another weakly nonlinear limit, called the **Bogoliubov limit**, where the system is described by a strong condensate with infinitesimal fluctuations



The dispersion relation for the perturbations is

$$\omega(\mathbf{k}) = \pm\,|\,\mathbf{k}\,|\sqrt{|\,\mathbf{k}\,| + 2\rho_\infty}$$

and 3-wave interactions

$$\frac{\partial n_1}{\partial t} = \int \left(\mathcal{R}^1_{2,3} - \mathcal{R}^3_{2,1} - \mathcal{R}^3_{1,2}\right) d\mathbf{k}_{23}\,, \quad \text{where}$$

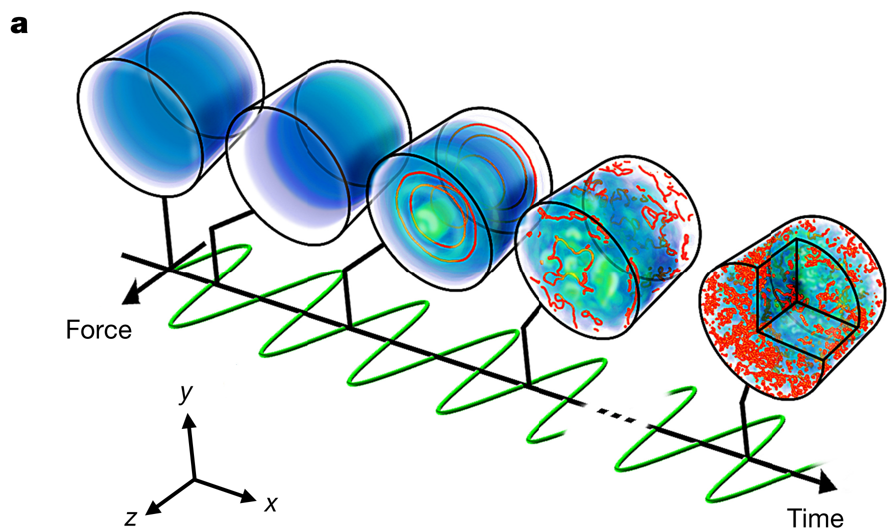
$$\mathcal{R}^3_{1,2} = \left| V^{\mathbf{k}_3}_{\mathbf{k}_1,\mathbf{k}_2} \right|^2 \delta(\omega_1 + \omega_2 - \omega_3)\,\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3)\,\left(n_1 n_2 - n_2 n_3 - n_3 n_1\right)$$

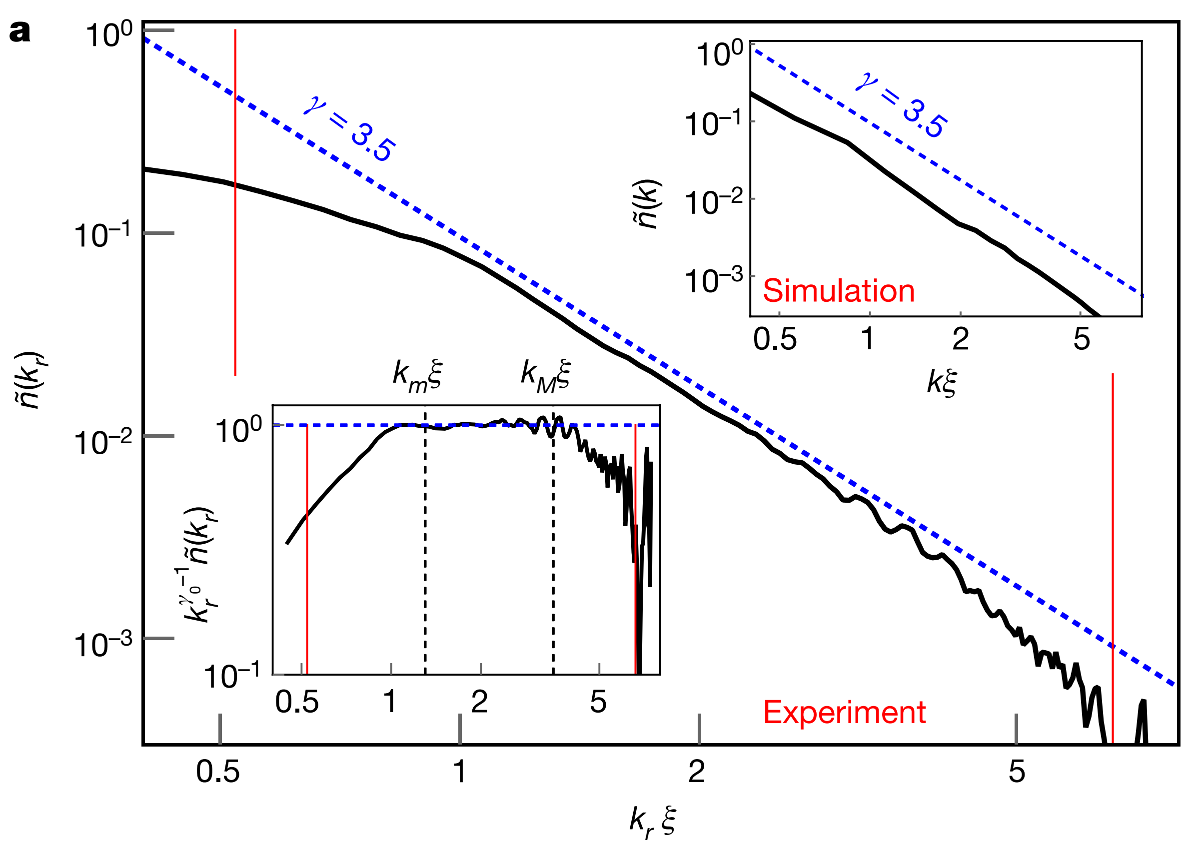
- In the limit of large scale fluctuations much larger than the healing length, the direct energy cascade results in $n^{(\psi)}_{1D} \propto k^{-7/2}$

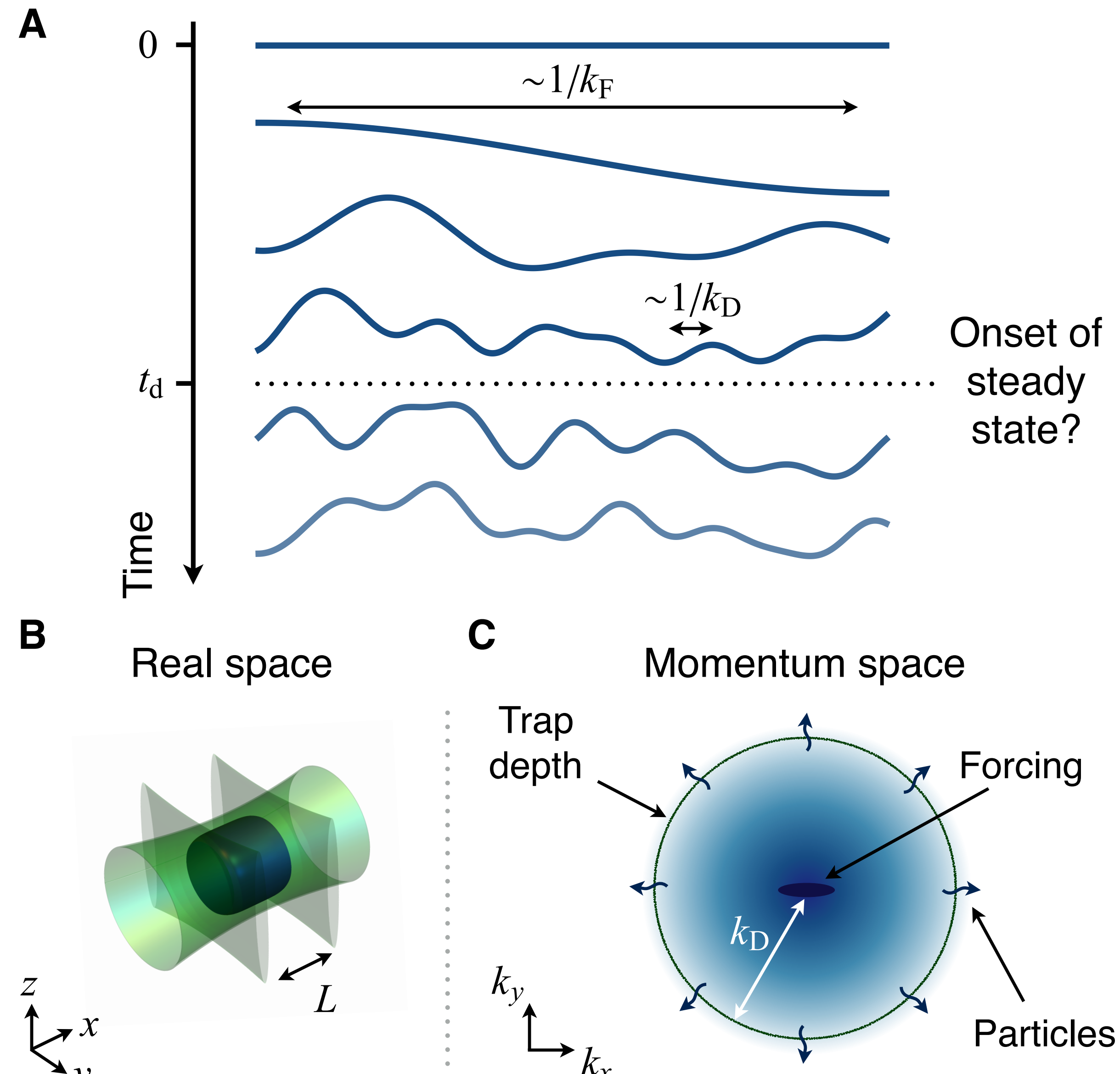
[Fujimoto & Tsubota, PRA 91, 053620 (2015)]

- Not completely clear how to derive the kinetic equation, as the dispersion relation is linear in ks and the resonant manifold is trivial

The interest is not only theoretical but also motivated by recent (3D) experiments in BECs







[Navon et al., Nature 532, 7627 (2016); Navon et al., Science 366, 382 (2019)]

THE KICKED-ROTOR TWO-DIMENSIONAL GP

We use the forced-dissipated 2D GP model

$$i\partial_t \psi + \nabla^2 \psi - |\psi|^2 \psi = \hat{\Gamma}_f \psi + \hat{\Gamma}_d \psi$$

The forcing operator represent a kick-rotor in the form

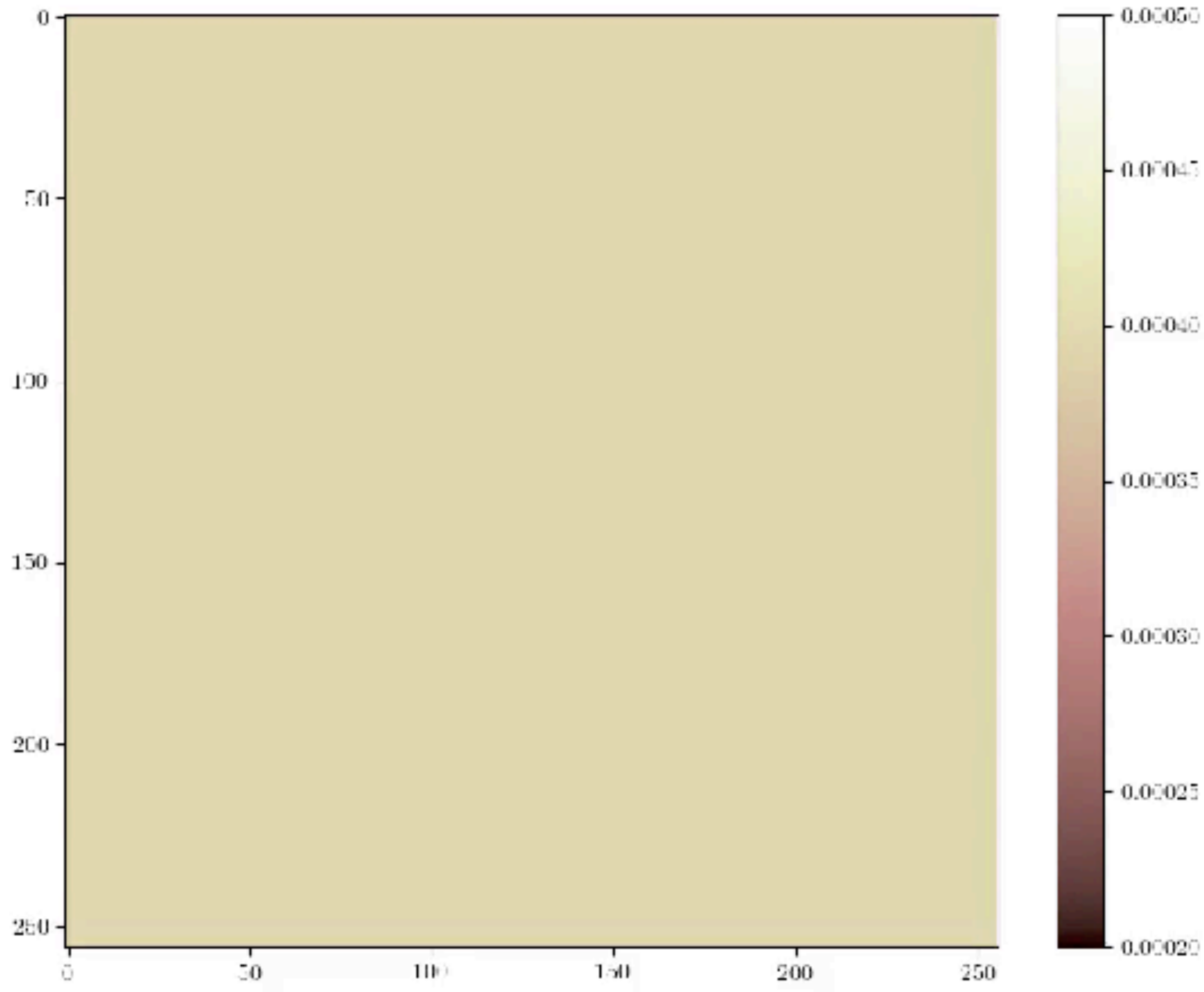
[Vermersch et al., arXiv:1410.2587; Garreu, private communication]

$$\hat{\Gamma}_f = K' \sum_{n \in \mathbb{N}} \left[\cos(k^\star x - \phi_{x,n}) + \cos(k^\star y - \phi_{y,n}) \right] \delta(t - nT)$$

- $\phi_{x,n}, \phi_{y,n} \in [0, 2\pi)$ are random variables uniformly distributed which shift the forcing at each nth kick
- Each kick is given after a period T

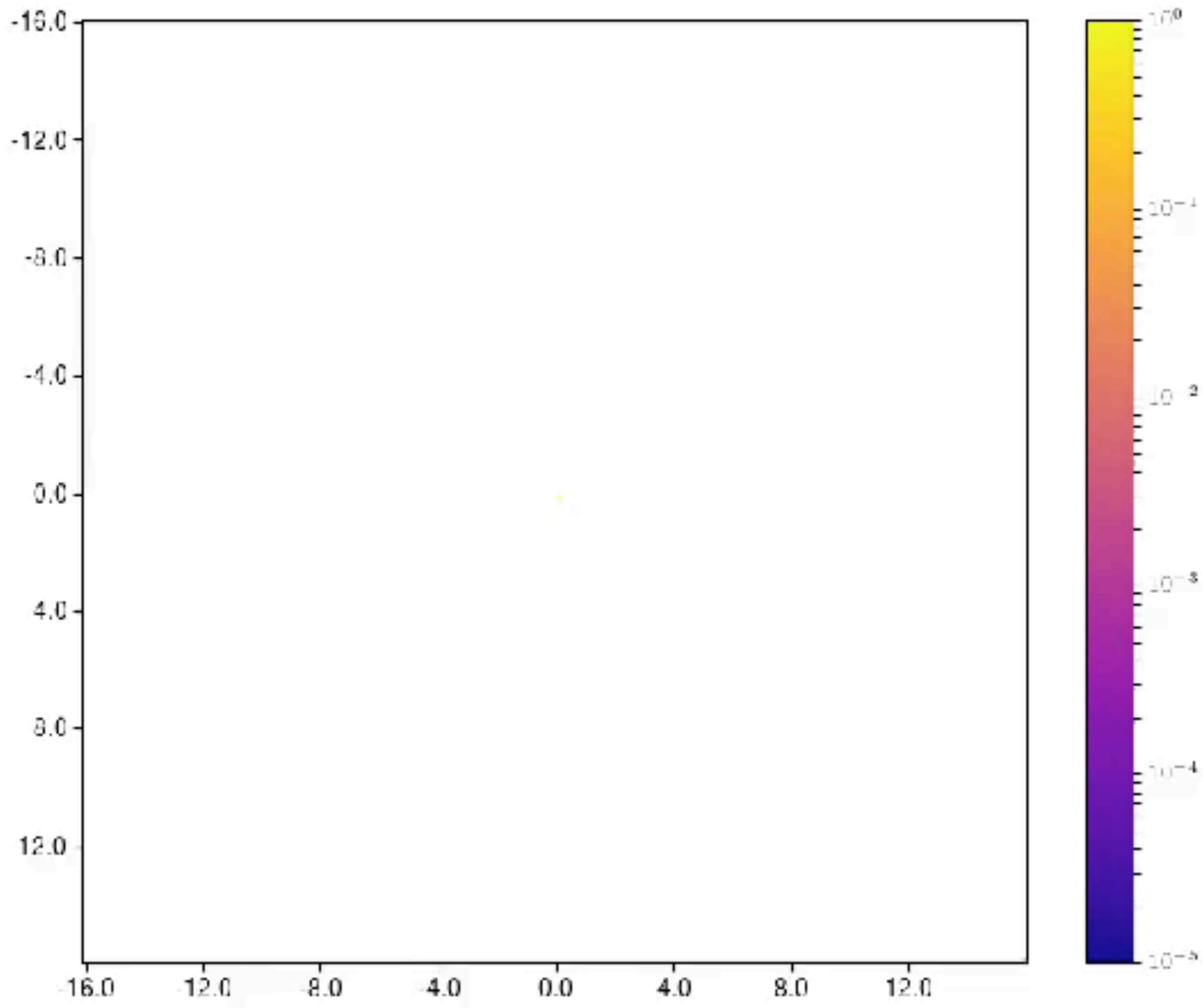
The dissipative operator $\hat{\Gamma}_d = i\nu \left(\nabla^2 \right)^3$ is a standard hyper-viscosity at small scales, it mimics a synthetic dissipation at small scales

DENSITY EVOLUTION, EXCITING THE 8TH MODE



Evolution of the density field $\rho = |\psi|^2$ in time. Here the system is forced at the 8th harmonic every $T \simeq 0.9\,\xi/c$. The ratio system size to healing length is $L/\xi = 256$.

SPECTRUM EVOLUTION, EXCITING THE 8TH MODE

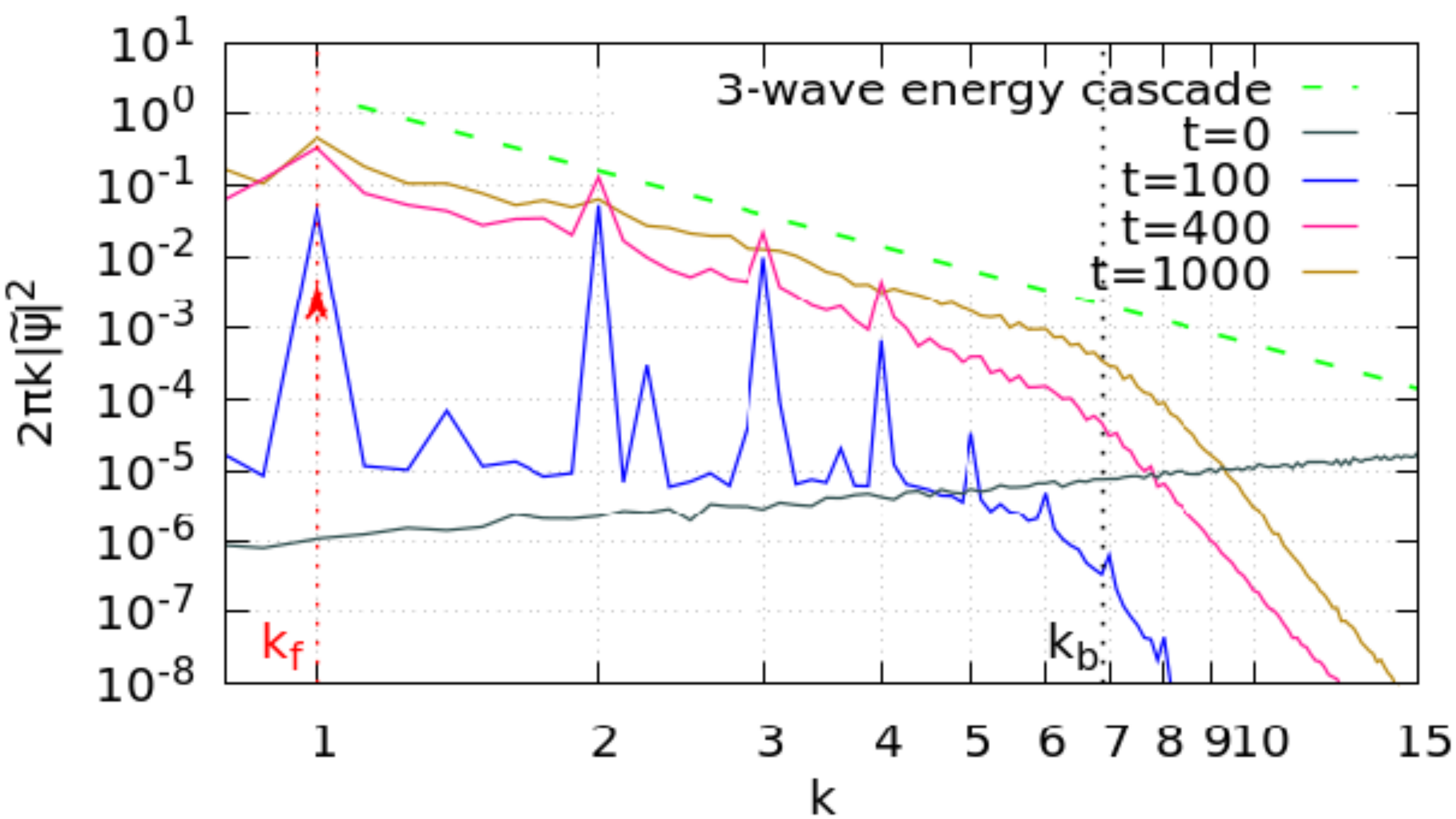


Evolution of the wave-action spectrum in time. Here the system is forced at the 8th harmonic every $T \simeq 0.9\,\xi/c$. The ratio system size to healing length is $L/\xi = 256$.

PHONON-INTERACTIONS AND CASCADE FORMATION

The dynamics has different regimes:

- discrete phonon-phonon interaction regime takes place: the energy is slowly transferred to small scales
- when the healing length/or the dissipative scales are populated, the phonons are able to interact with the free particle-like excitations
- the energy finally spreads over all the accessible modes of the system (decay from quasi-solitons to dipoles)

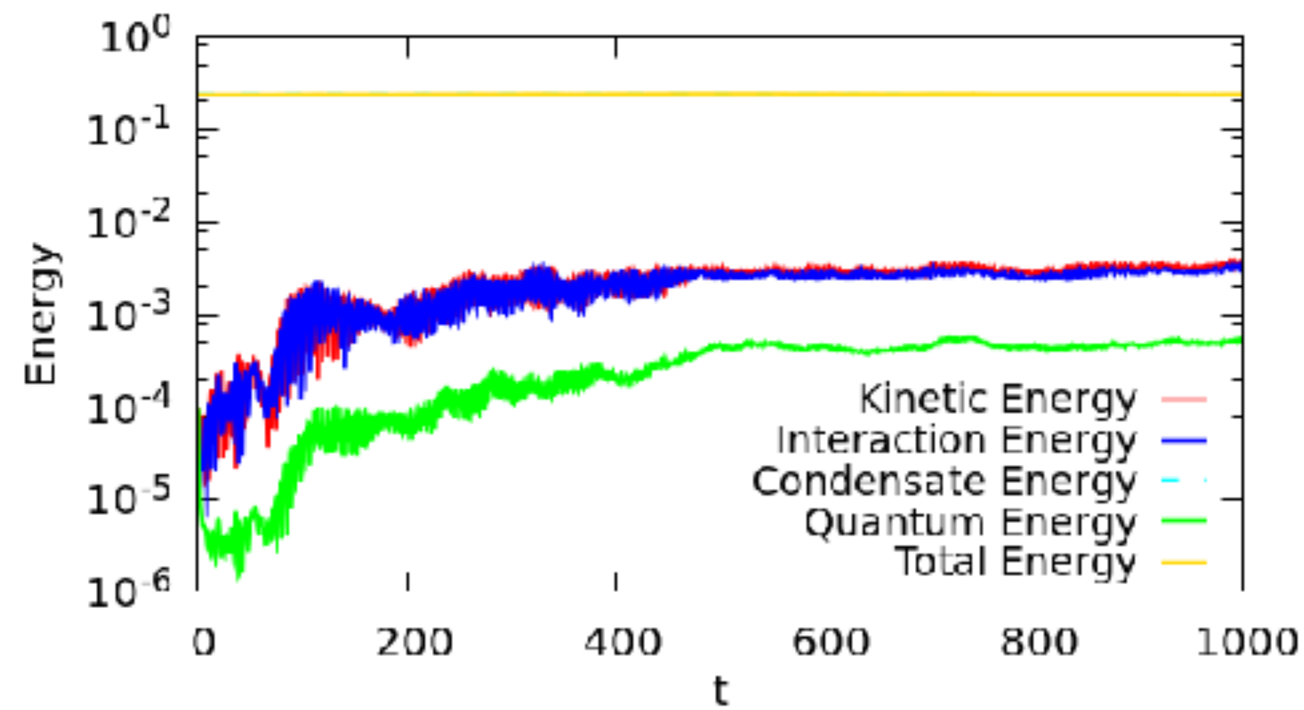


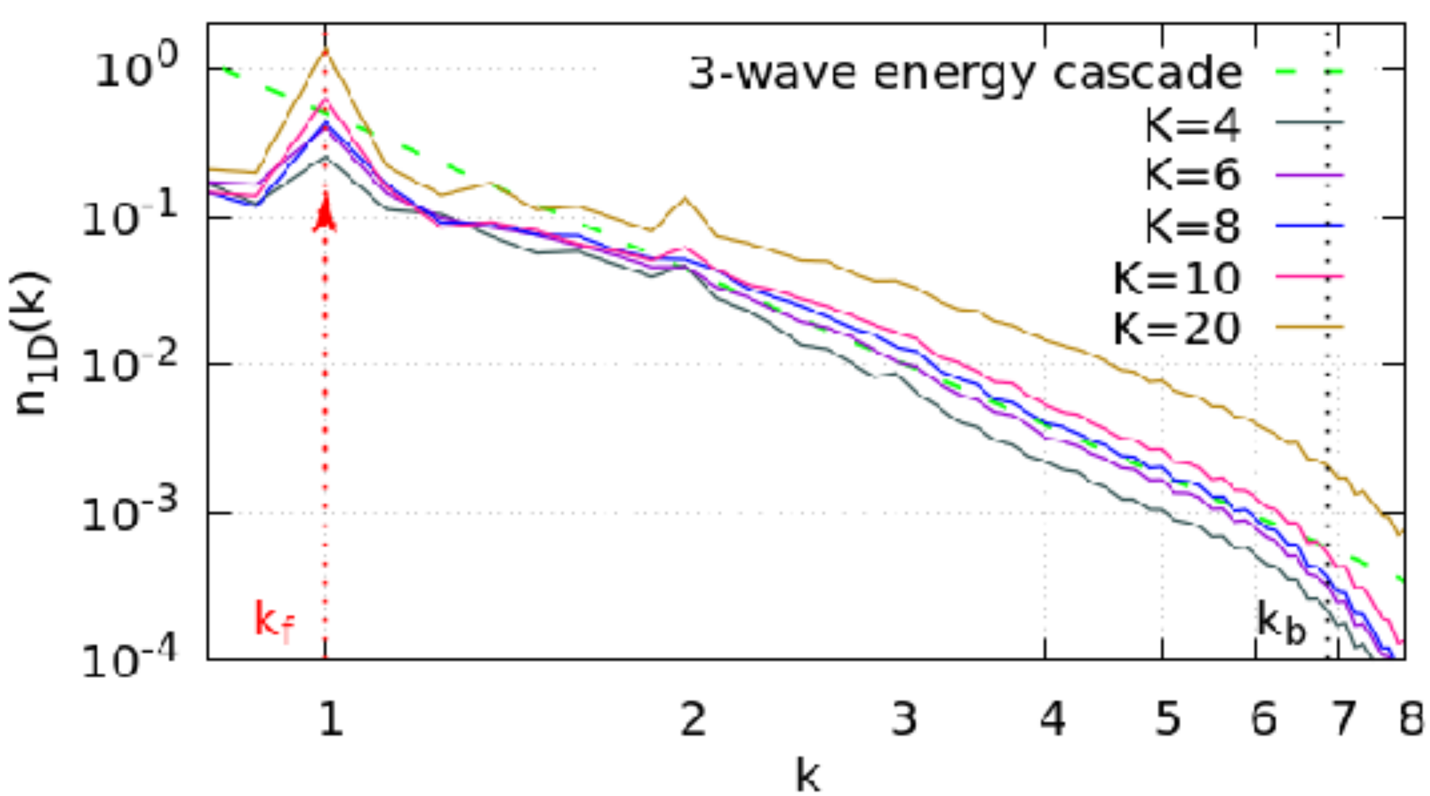
- a power-law with exponent comparable with the one predicted by the WT theory is observed in the wave-action spectrum

$$n_{1D}^{(\psi)} \propto k^{-7/2}$$

ROBUSTNESS OF THE WT BOGOLIUBOV CASCADE

- In the statistically steady state the linear energy density is two order of magnitude smaller compared to the nonlinear one, so the Bogoliubov regime should still be valid

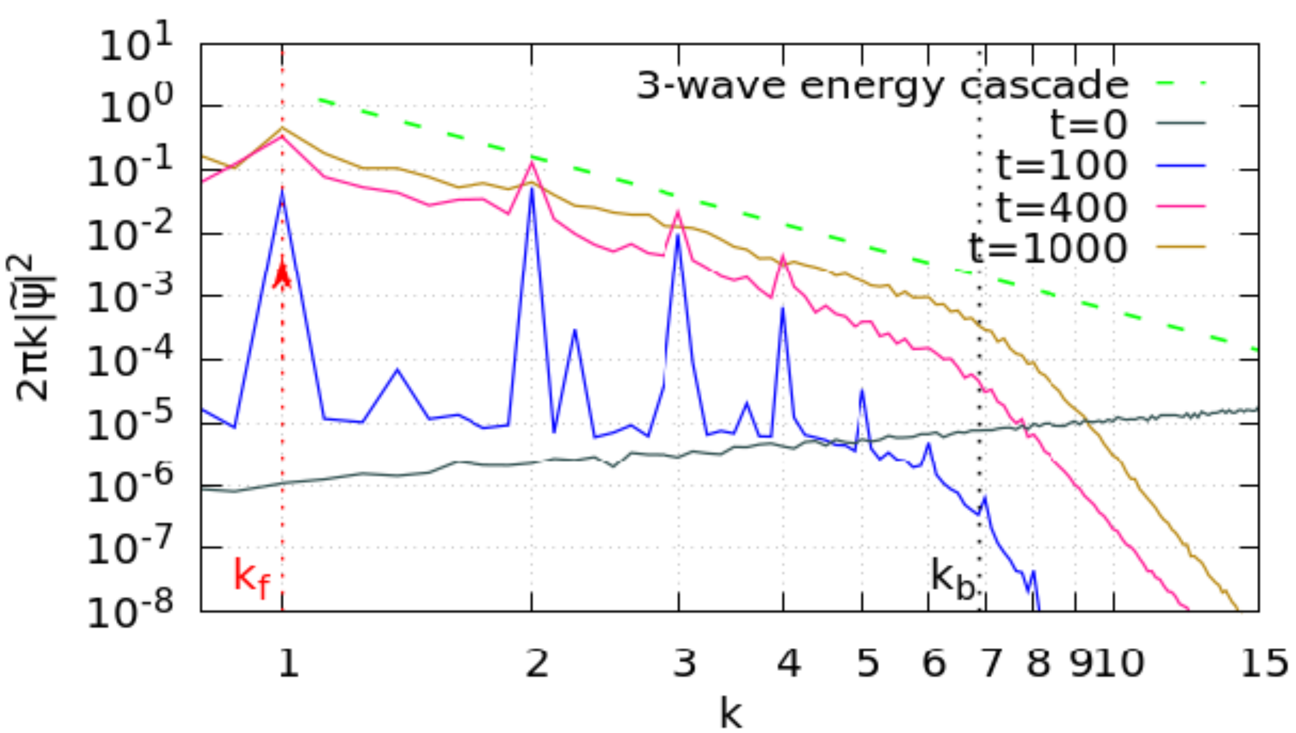




- For different forcing amplitudes the spectra are close to the WT prediction, except when the amplitude becomes very large and the spectrum flattens a bit

CONCLUSIONS

- This system shows an interesting interplay between discrete phonon wave-interactions and denser wave turbulence
- Phonons spectra are not initially captured by the WT theory
- Dispersive free-particle excitations trigger the re-distribution of energy towards all the accessible modes of the system
- Finally a WT cascade seems to take place



- This system could in principle be accurately realised in two-dimensional BEC experiments
- What happens when non-local interactions are considered? Helium?

THANKS FOR YOUR ATTENTION!