STOKES DRIFT AND IMPURITY TRANSPORT IN A QUANTUM FLUID

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STOKES DRIFT AND IMPURITY TRANSPORT IN A QUANTUM FLUID

- Recap on Stokes drift in classical fluids
- Gross-Pitaevskii model with classical impurities
- Transport of a single impurity in 2d

THE STOKES DRIFT

"The Stokes drift velocity is the average velocity when following a specific fluid parcel as it travels with the fluid flow."

[Quote and picture below taken from "Stokes drift" on Wikipedia]

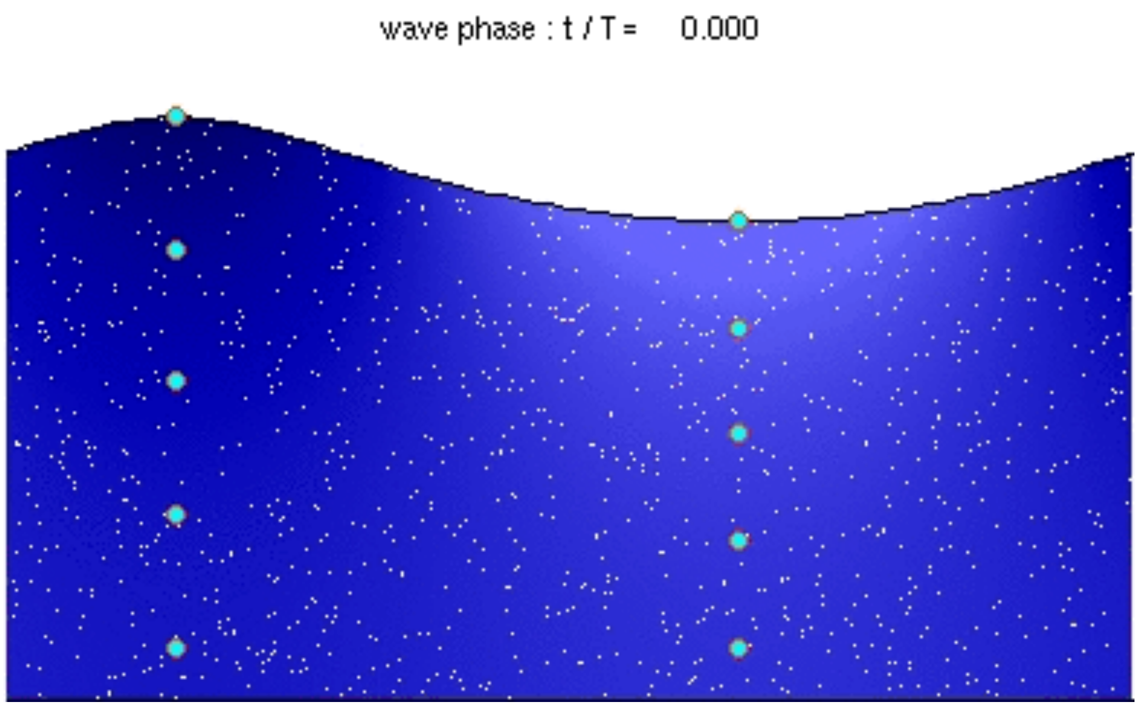
THE STOKES DRIFT

This effect was derived first by G.G. Stokes in 1847 in the context of surface gravity waves

[Stokes, TCPS 8, 441, 1847; Longuet-Higgins, PTRSA 245, 535 (1953)]

- Go from Eulerian to Lagrangian framework
- Multiple-time scale expansion
- For a sinusoidal wave $\eta = a\cos(kx - \omega t)$, at the second order in the wave steepness the drift results in

$$v_{\mathrm{drift}} = \omega k a^2 e^{2kz}$$



[Deep_water_wave.gif,
from "Stokes drift" on Wikipedia]

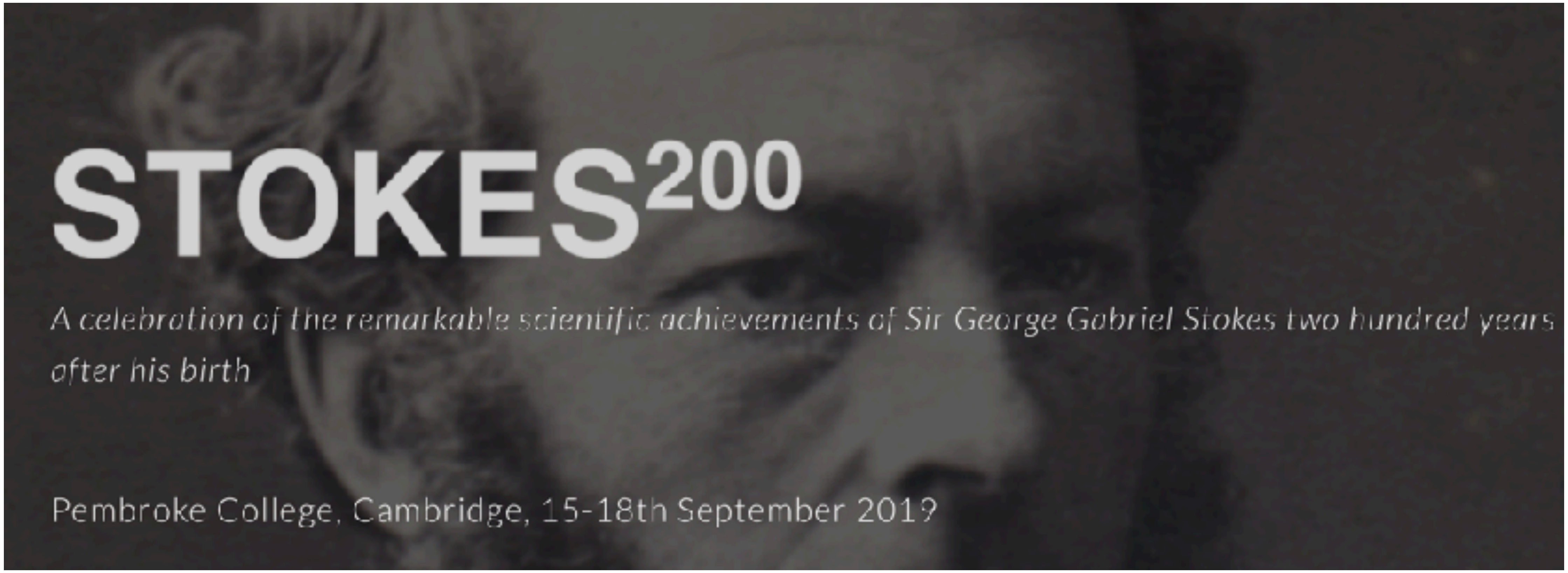
WHAT IF WE CONSIDER PARTICLES IMMERSED IN THE FLUID?

- A similar derivation can be repeated to study the motion of an external particle, once defining its equation of motion depending on the fluid velocity is postulated

$$\ddot{\mathbf{q}} = F(\mathbf{u}, \ldots)$$

- Perfect tracers: small particles with infinite Stokes drag
- Buoyancy or other inertial effects

[Longuet-Higgins, PTRSA 245, 535 (1953); Santamaria et al., EPL 102, 14003 (2013)]



[https://stokes200.weebly.com]

IS THERE AN ANALOGUE OF STOKES DRIFT IN A QUANTUM FLUID (INVISCID AND COMPRESSIBLE)?

IF SO, HOW DOES IT AFFECT THE TRANSPORT OF PARTICLES (IMPURITIES)?

THE GROSS-PITAEVSKII MODEL

$$\imath\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi-g\,|\,\psi\,|^2\psi-V_{ext}\psi=0$$

- It is a mean-field equation that can be formally derived to model dilute Bose gases in the limit of zero temperature
- It also model qualitatively well other superfluids like liquid Helium below the λ-point
- This model is nothing but a nonlinear Schroedinger equation, where $\psi(\mathbf{r},t)$ is a complex function describing the order parameter of the system
- m is the mass of each boson, $\hbar$ is the reduced Planck's constant, g weight the effective binary collisions between the bosons, V_{ext} is some external potential

THE GROSS-PITAEVSKII MODEL

$$\imath\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi-(g\,|\,\psi\,|^2-\mu)\psi-V_{ext}\psi=0$$

Using Madelung transformation $\psi=\sqrt{\rho}\,\exp(\imath\phi)$ and defining density and velocity as $\rho=m\,|\,\psi\,|^2$ and $\mathbf{v}=\hbar/m\nabla\phi$, respectively, then

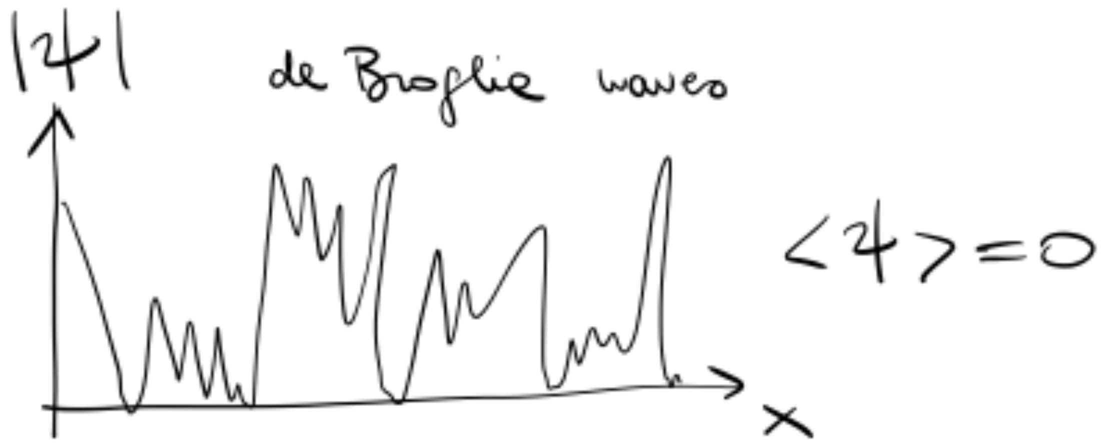
$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$$

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\nabla\left[-\frac{g}{m}\left(\rho-\frac{\mu}{g}\right)+\frac{1}{m}V_{ext}+\frac{\hbar^2}{2m^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right]$$

- The GP models an inviscid, barotropic, and irrotational fluid
- The last term of the second equation is the quantum pressure

TWO WEAKLY NONLINEAR LIMITS IN THE GP MODEL

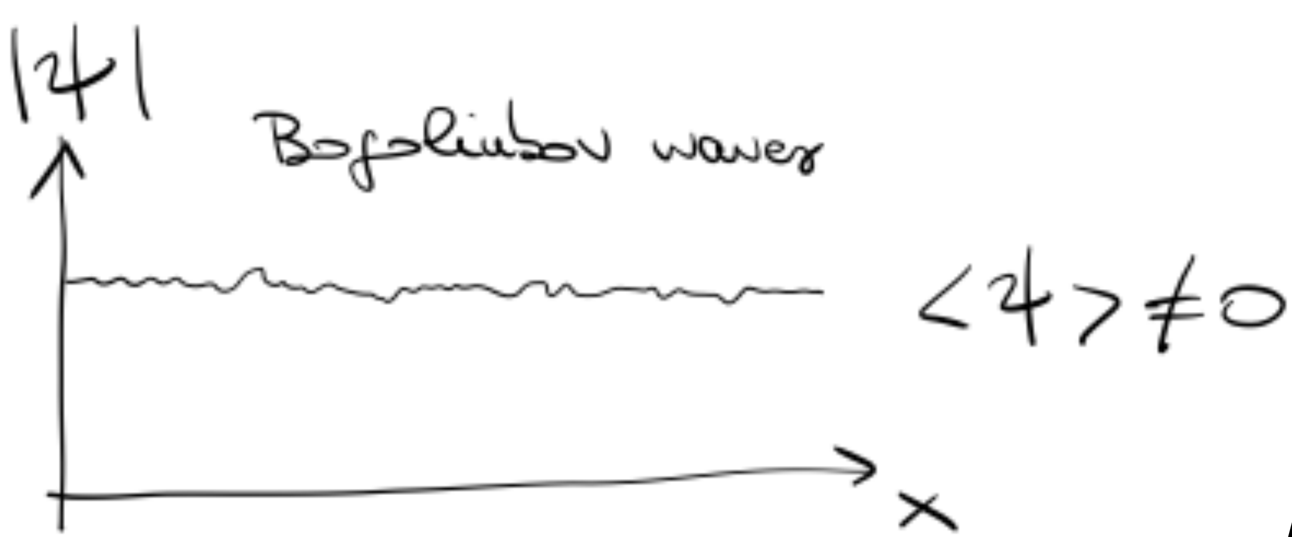
- The **de Broglie limit** is the limit where no modes are macroscopically occupied (no strong condensate)



The dispersion relation for the waves is

$$\omega(\mathbf{k}) = \frac{\hbar^2}{2m}|\mathbf{k}|^2$$

- The **Bogoliubov limit**, where the system is described by a strong condensate with small density/phase fluctuations on top



The dispersion relation for the perturbations is

$$\omega(\mathbf{k}) = \pm\frac{\hbar}{\sqrt{2m}}|\mathbf{k}|\sqrt{\frac{\hbar^2}{2m}|\mathbf{k}| + \frac{2g\rho_0}{m}}$$

$$\rho_0 = \frac{\mu}{g}, \quad \xi = \sqrt{\hbar^2/2g\rho_0}, \quad c = \sqrt{g\rho_0/m^2}$$

THE GP MODEL WITH CLASSICAL IMPURITIES

We introduce active impurities in the GP model by considering them as classical-like particles with position and momentum $(\mathbf{q}_i, \mathbf{p}_i)$ and identical masses M_{p}.

[Winiecki & C. Adams, EPL 52, 257 (2000); Shukla et al., PRA 94, 041602 (2016); Shukla et al., PRA 97, 013627 (2018)]

$$\begin{cases} \imath\hbar\dfrac{\partial\psi}{\partial t} = -\dfrac{\hbar^2}{2m}\nabla^2\psi + (g\,|\psi|^2 - \mu)\psi + \displaystyle\sum_{i=1}^{N_{\mathrm{p}}} V_{\mathrm{p}}(|\,\mathbf{x}-\mathbf{q}_i\,|)\psi \\ M_{\mathrm{p}}\ddot{\mathbf{q}}_i = -\displaystyle\int V_{\mathrm{p}}(|\,\mathbf{x}-\mathbf{q}_i\,|)\,\nabla\,|\psi|^2\,d\mathbf{x} - \sum_{i\neq j}^{N_{\mathrm{p}}} \nabla V_{rep}(|\,\mathbf{q}_i - \mathbf{q}_j\,|) \end{cases}$$

This model naturally conserves the number of impurities N_I, the energy (the Hamiltonian), the number of bosons and the total momentum:

$$H = \int \frac{\hbar^2}{2m}|\nabla\psi|^2 + \frac{g}{2}|\psi|^4 + \sum_{i=1}^{N_{\mathrm{p}}} V_{\mathrm{p}}(|\,\mathbf{x}-\mathbf{q}\,|)\,|\psi|^2\,d\mathbf{x} + \sum_{i=1}^{N_{\mathrm{p}}}\frac{\mathbf{p}_i^2}{2M_{\mathrm{p}}} + \frac{1}{2}\sum_{i<j}^{N_{\mathrm{p}}} V_{rep}(|\,\mathbf{q}_i-\mathbf{q}_j\,|)$$

$$N = \int |\psi|^2\,d\mathbf{x} \quad \text{and} \quad \mathbf{P} = \imath\frac{\hbar}{2}\int\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)\,d\mathbf{x} + \sum_{i=1}^{N_{\mathrm{p}}}\mathbf{p}_i$$

THE GP MODEL WITH CLASSICAL IMPURITIES

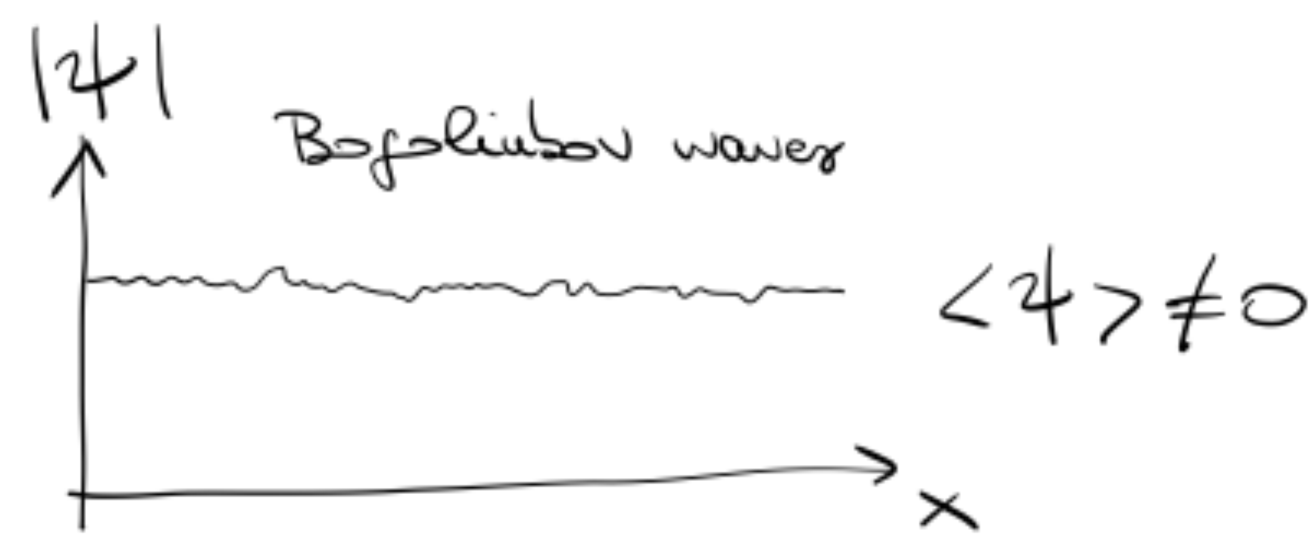
If a single impurity is considered, the model reads simply

$$\begin{cases} \imath\hbar\dfrac{\partial\psi}{\partial t} = -\dfrac{\hbar^2}{2m}\nabla^2\psi + (g\,|\,\psi\,|^2 - \mu)\psi + V_{\mathrm{p}}(\,|\,\mathbf{x}-\mathbf{q}\,|\,)\psi \\ M_{\mathrm{p}}\ddot{\mathbf{q}} = -\displaystyle\int V_{\mathrm{p}}(\,|\,\mathbf{x}-\mathbf{q}\,|\,)\,\nabla\,|\,\psi\,|^2\,d\mathbf{x} \end{cases}$$

- We imagine an impurity of size comparable with the healing length ξ
- The Bogoliubov density/phase waves, written in the fluid dynamical framework are

$$\rho = \rho_0 + A_\rho \cos(kx - \omega t)$$

$$v_{\mathrm{w}} = \frac{\omega A_\rho}{k\rho_0}\cos(kx - \omega t)$$

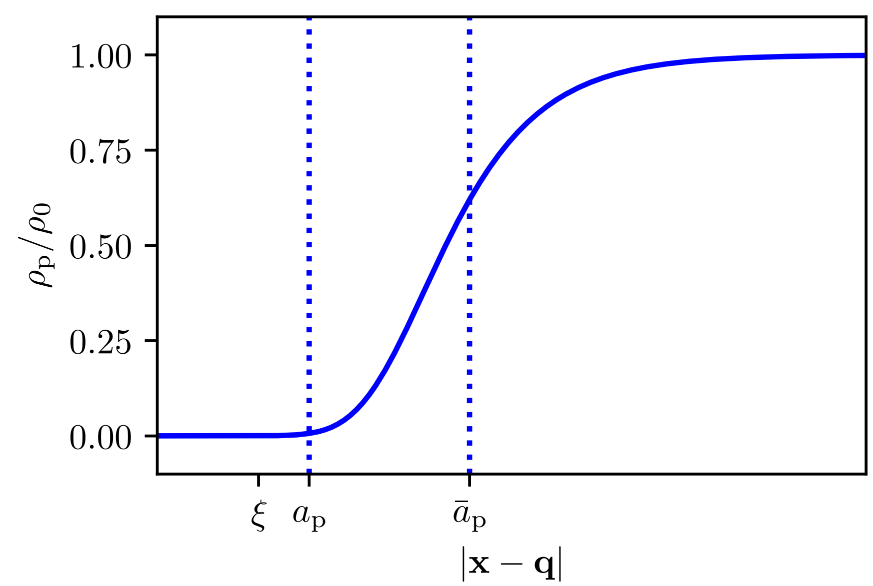


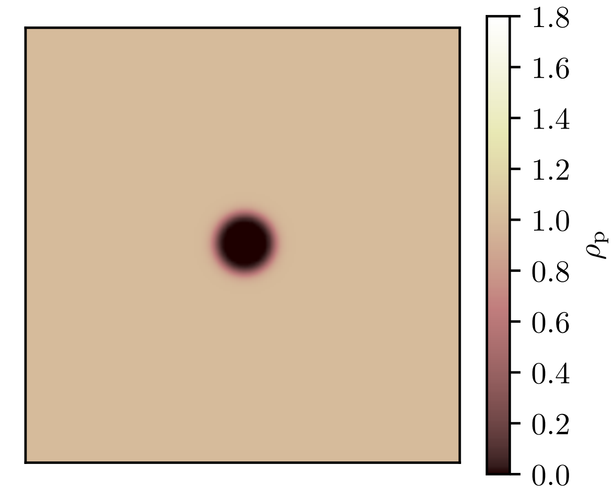
THE GP MODEL WITH CLASSICAL IMPURITIES

The quantum fluid-impurity interaction is modelled using a phenomenological hat-shaped impurity potential

$$V_{\mathrm{p}}(r) = \frac{V_0}{2}\left[1 - \tanh\left(\frac{r^2 - \eta_a^2}{4\Delta_a^2}\right)\right]$$

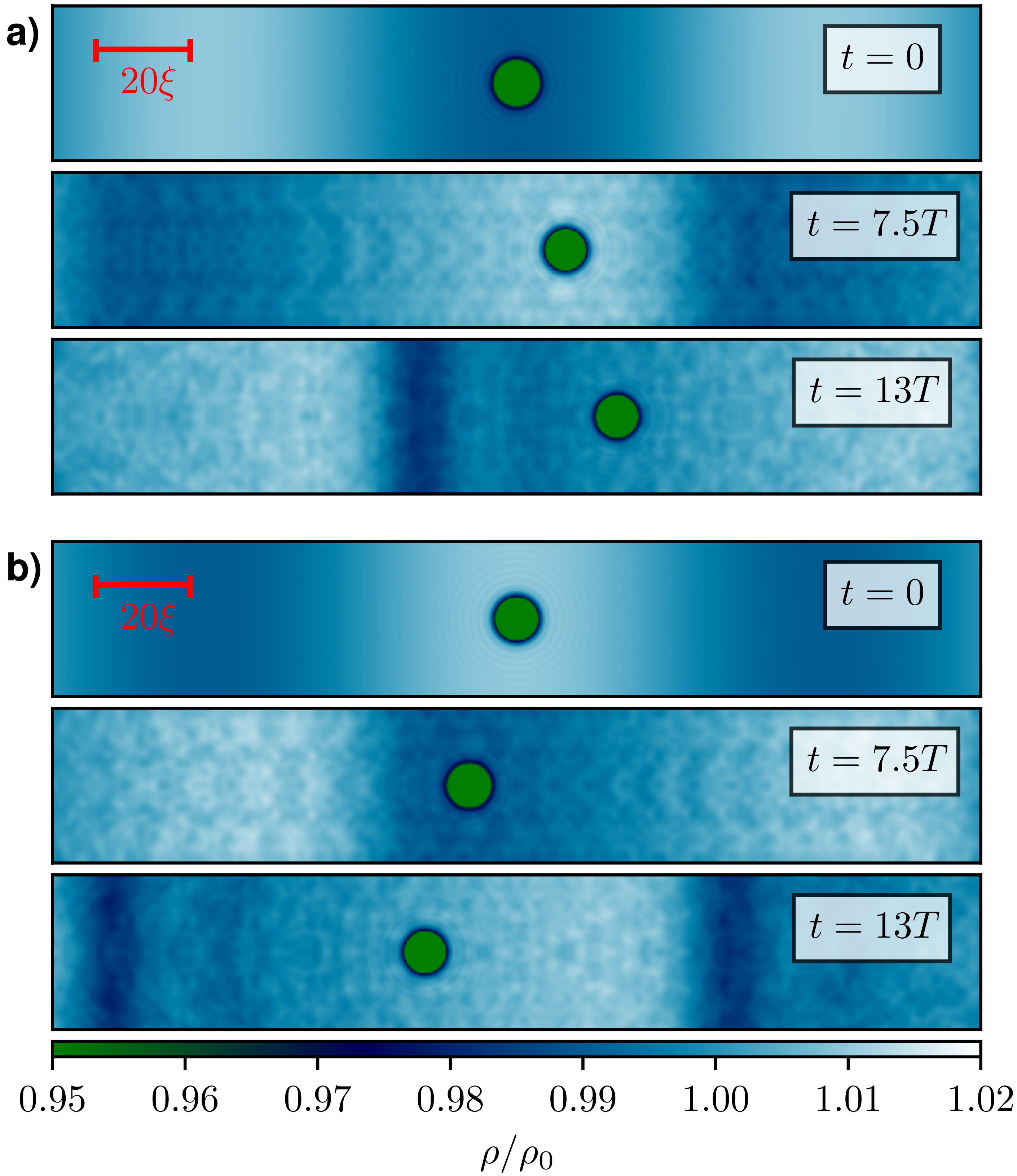
[Shukla et al., PRA 94, 041602 (2016);
Giuriato et al., JPA 52, 305501 (2019);
Giuriato & Krstulovic, SciRep 9, 4839 (2019)]





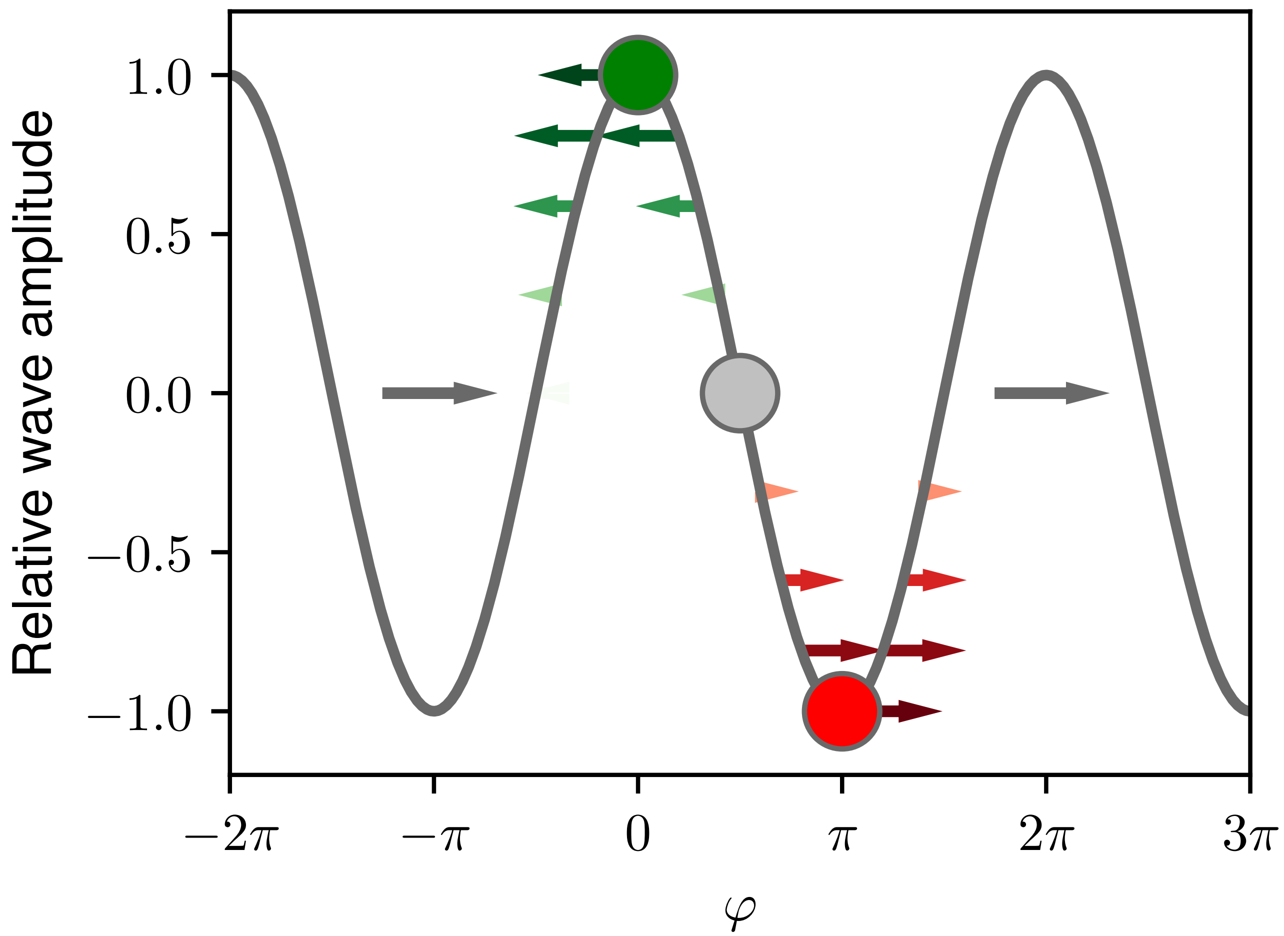
- the quantum fluid heals at distance order of ξ
- an effective particle radius $\bar{a}_{\mathrm{p}} > a_{\mathrm{p}}$ is estimated by measuring the volume of the displaced fluid $\pi \bar{a}_{\mathrm{p}}{}^2 = \int (|\psi_0|^2 - |\psi_{\mathrm{p}}|^2)\,\mathrm{d}\mathbf{x}$, here ψ_{p} is the steady state with one impurity
- non-dimensional impurity mass as $\mathscr{M} = M_{\mathrm{p}}/M_0$, where $M_0 = \rho_0 \pi \bar{a}_{\mathrm{p}}{}^2$
- we set $V_0 = 20\mu$, $a_{\mathrm{p}} = 1.5\xi$, $\Delta_a = 0.75\xi$ and $\eta_a = \xi$, leading to $\bar{a}_{\mathrm{p}} = 3.1\xi$

AN IMPURITY HIT BY A DENSITY/PHASE WAVE



- The impurity is initially steady
- The motion depends on the initial impurity-wave phase ($\varphi = 0$ when the impurity is at the wave crest)
- Only a smaller fraction of the computational domain is shown here

AN IMPURITY HIT BY A DENSITY/PHASE WAVE



Sketch of the initial motion of the impurity versus the impurity wave phase φ: the impurity moves towards the region of lower pressure!

- The impurity is initially steady
- The motion depends on the initial impurity-wave phase ($\varphi = 0$ when the impurity is at the wave crest)
- Only a smaller fraction of the computational domain is shown here

STOKES DRIFT, THEORETICAL PREDICTIONS (1/2)

- Impurity is a ball of radius $\bar{a}_\mathrm{p}$
- Passive particle in first approximation, $\rho = \rho_\mathrm{p}(\rho_0 + \rho_\mathrm{w})/\rho_0$
- Small impurity compared to the density/phase wave, $ka_\mathrm{p} \ll 1$

$$M_\mathrm{p}\ddot{\mathbf{q}} = \mathbf{F} = -\int V_\mathrm{p}(|\mathbf{x}-\mathbf{q}|)\nabla|\psi|^2\,\mathrm{d}\mathbf{x} \quad \Longrightarrow \quad \mathbf{M}_\mathrm{p}\ddot{\mathbf{q}} \simeq \gamma_2\mathbf{C}_\mathrm{a}\mathbf{M}_0\left(\frac{\mathrm{d}\mathbf{v}_\mathrm{w}}{\mathrm{d}\mathbf{t}}\bigg|_\mathbf{q} - \ddot{\mathbf{q}}\right) + \gamma_1\mathbf{M}_0\frac{\mathrm{d}\mathbf{v}_\mathrm{w}}{\mathrm{d}\mathbf{t}}\bigg|_\mathbf{q}$$

Where we have introduced the added mass coefficient ($C_\mathrm{a} = 1$ in 2D) and two phenomenological dimensionless parameters $\gamma_1 \simeq 0.69$ and $\gamma_2 \simeq 0.25$ which account for the presence of a healing layer at the particle boundary (values were obtained by fitting).

The impurity dynamics is driven by the effective equation

$$\ddot{q} = \epsilon\frac{\omega^2}{k}\sin(kq - \omega t)\,, \quad \text{where}$$

$$\epsilon = \eta\frac{A_\rho}{\rho_0}, \qquad \text{with} \quad \eta = \left(\frac{\gamma_2 C_\mathrm{a} + \gamma_1}{\gamma_2 C_a + \mathscr{M}}\right),$$

STOKES DRIFT, THEORETICAL PREDICTIONS (2/2)

The impurity dynamics is driven by the effective equation

$$\ddot{q} = \epsilon \frac{\omega^2}{k} \sin(kq - \omega t)\,, \quad \text{where}$$

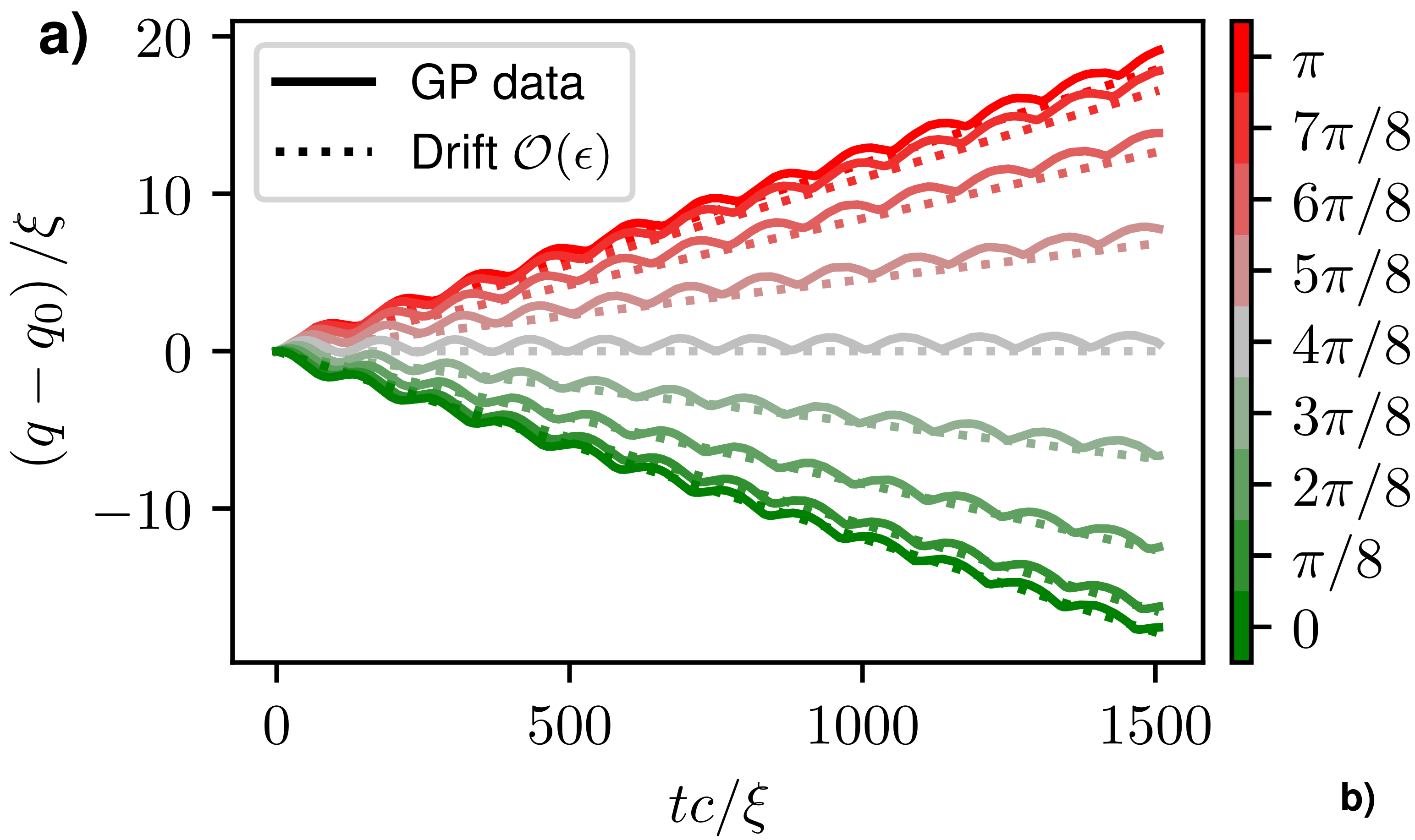
$$\epsilon = \eta \frac{A_\rho}{\rho_0}, \qquad \text{with} \quad \eta = \left(\frac{\gamma_2 C_\mathrm{a} + \gamma_1}{\gamma_2 C_a + \mathscr{M}} \right),$$

- Multiple-time scale expansion $q(t, \epsilon) = Q(t, \tau, \epsilon)$, given $\tau = \epsilon t$
- After averaging over the fast timescale t, the solution up to the second order is

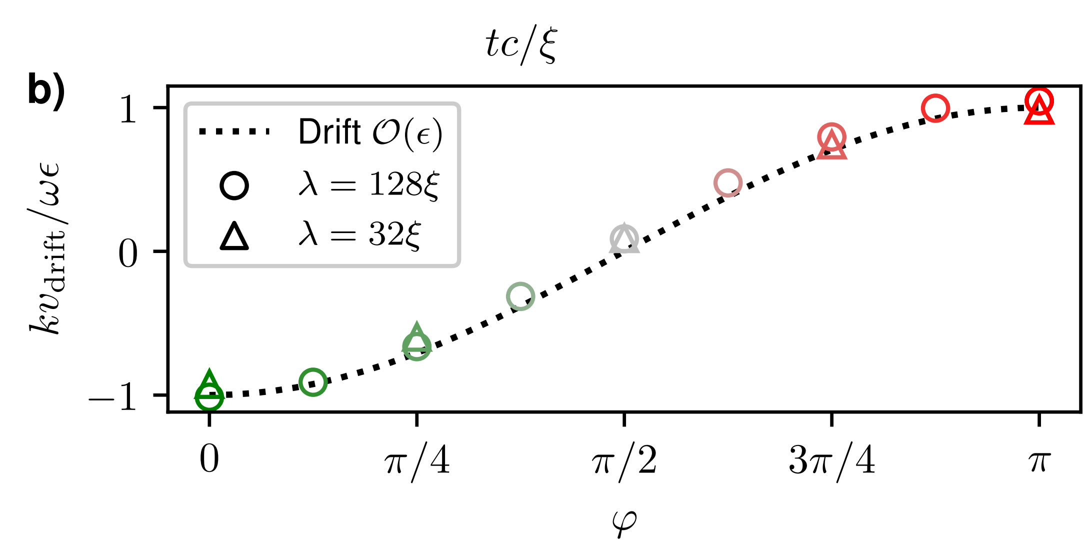
$$v_\mathrm{drift} = \langle \dot{q} \rangle_t = -\frac{\omega}{k}\epsilon \cos(\varphi) + \frac{\omega}{k}\epsilon^2 \left(1 + \frac{1}{4}\cos(2\varphi) \right) + \mathcal{O}(\epsilon^3)$$

STOKES DRIFT, COMPARISON WITH SIMULATIONS

$$v_{\text{drift}} = \langle \dot{q} \rangle_t = -\frac{\omega}{k}\epsilon\cos(\varphi) + \frac{\omega}{k}\epsilon^2\left(1 + \frac{1}{4}\cos(2\varphi)\right) + \mathcal{O}(\epsilon^3)$$



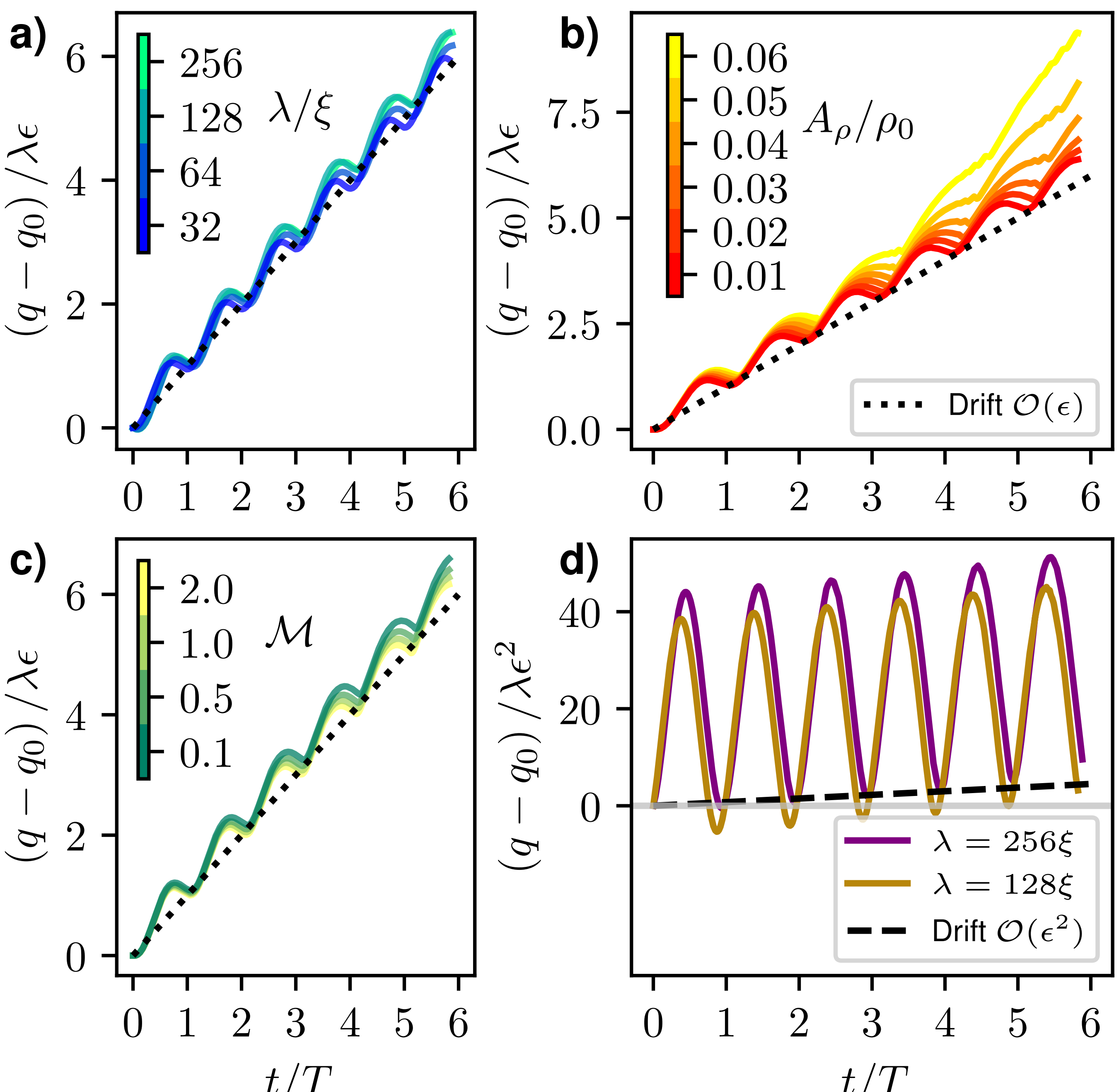
a) Time evolution of the impurity rescaled position (solid lines) for different impurity-wave phases. Dotted lines represent the drift prediction at the leading order.



b) Rescaled drift versus the impurity-wave phases for waves of wavelength $\lambda = 128\xi$ (circles) and $\lambda = 32\xi$ (triangles); the dotted line is the prediction at the leading order.

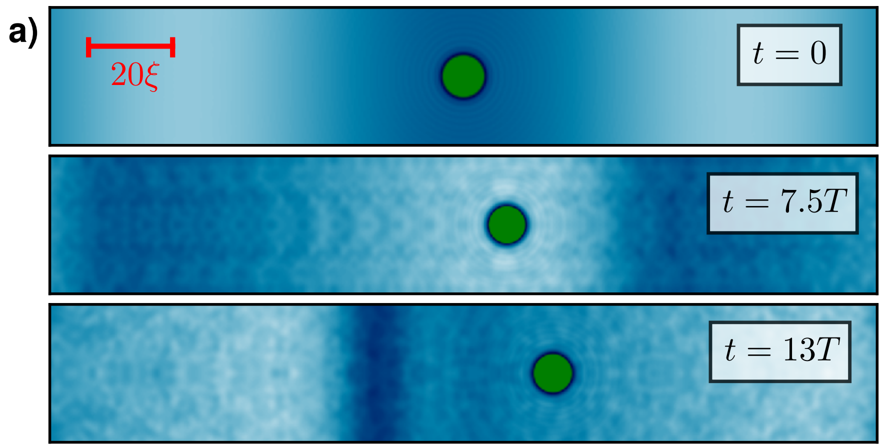
STOKES DRIFT, COMPARISON WITH SIMULATIONS

$$v_{\mathrm{drift}} = \langle \dot{q} \rangle_t = -\frac{\omega}{k}\epsilon\cos(\varphi) + \frac{\omega}{k}\epsilon^2\left(1 + \frac{1}{4}\cos(2\varphi)\right) + \mathcal{O}(\epsilon^3)$$

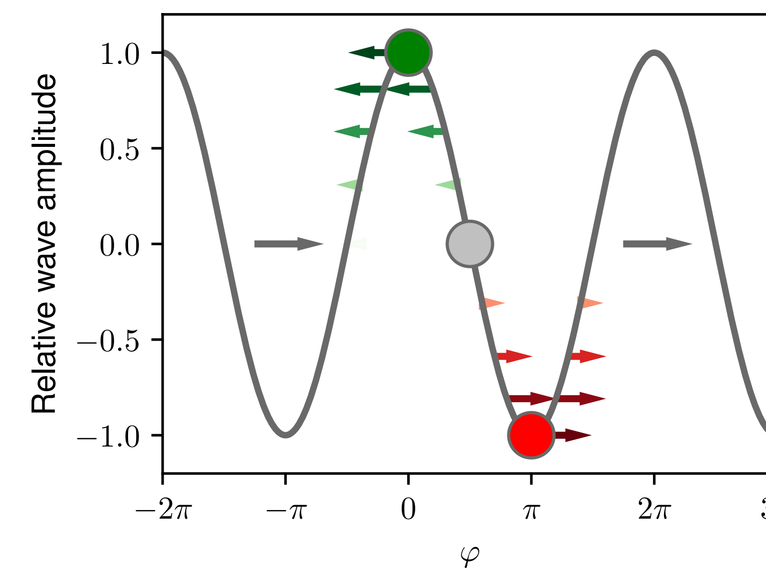


Time evolution of the impurity rescaled position with drift parameter ϵ for a) waves of different wavelength, b) waves of different amplitude and c) impurities of different mass. Dotted lines represent the drift prediction at the leading order. d) Time evolution of the impurity rescaled position with drift parameter ϵ^2 for waves of different wavelengths and same initial impurity-wave phase $\varphi = \pi/2$; the second order prediction is displayed in dashed line.

SUMMARY AND CONCLUSIONS



- A classical impurity is transported by an inviscid quantum fluid due to density fluctuations

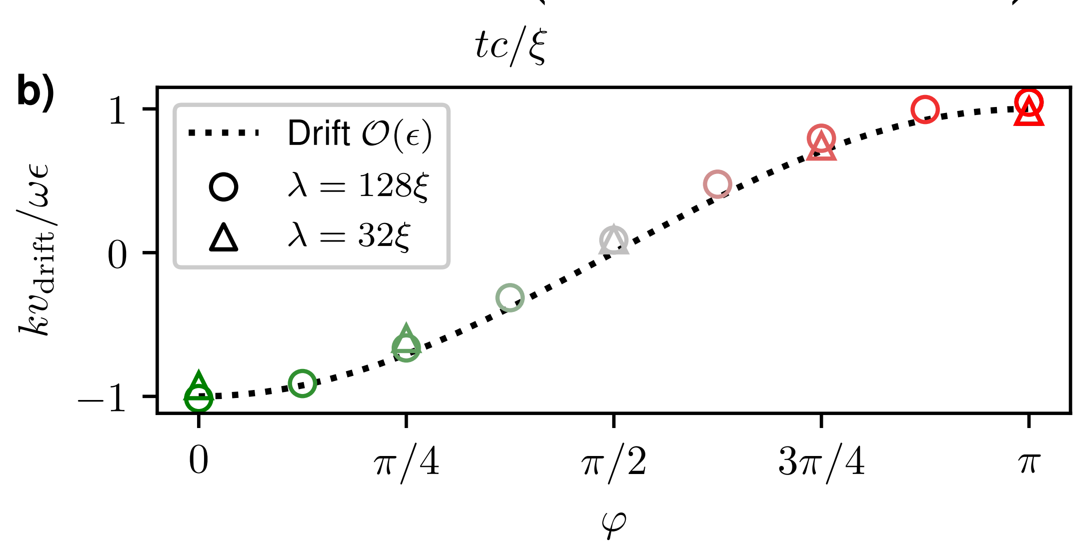


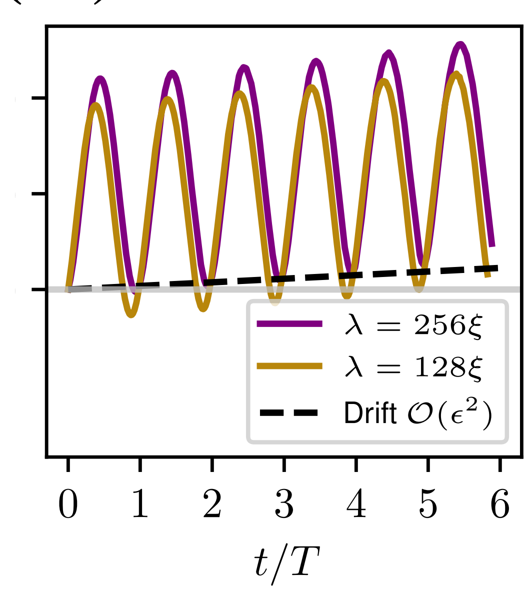
- The direction of the motion depends on the initial impurity-wave phase

- The Stoked drift prediction reads

$$v_{\text{drift}} = \langle \dot{q} \rangle_t = -\frac{\omega}{k}\epsilon \cos(\varphi) + \frac{\omega}{k}\epsilon^2 \left(1 + \frac{1}{4}\cos(2\varphi)\right) + \mathcal{O}(\epsilon^3)$$

- It is accurate at the first two orders





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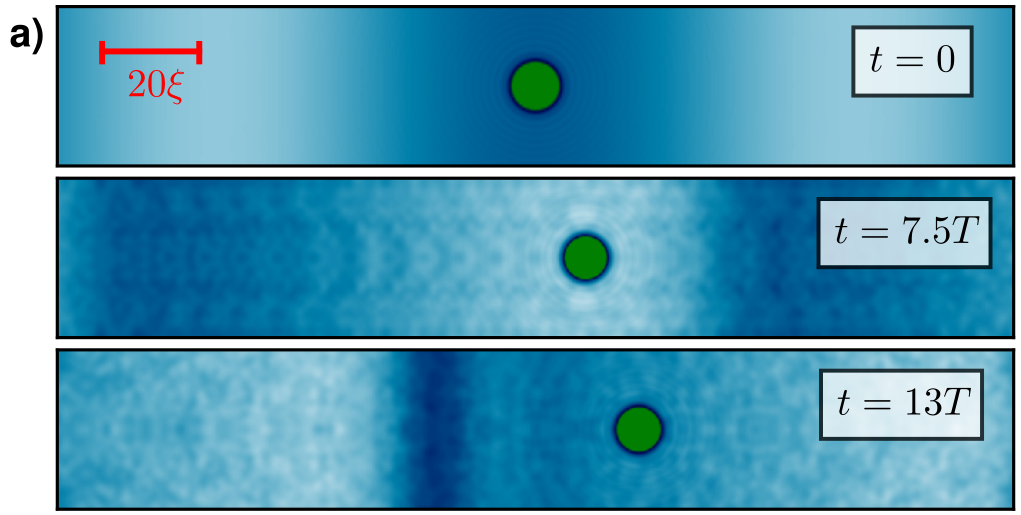
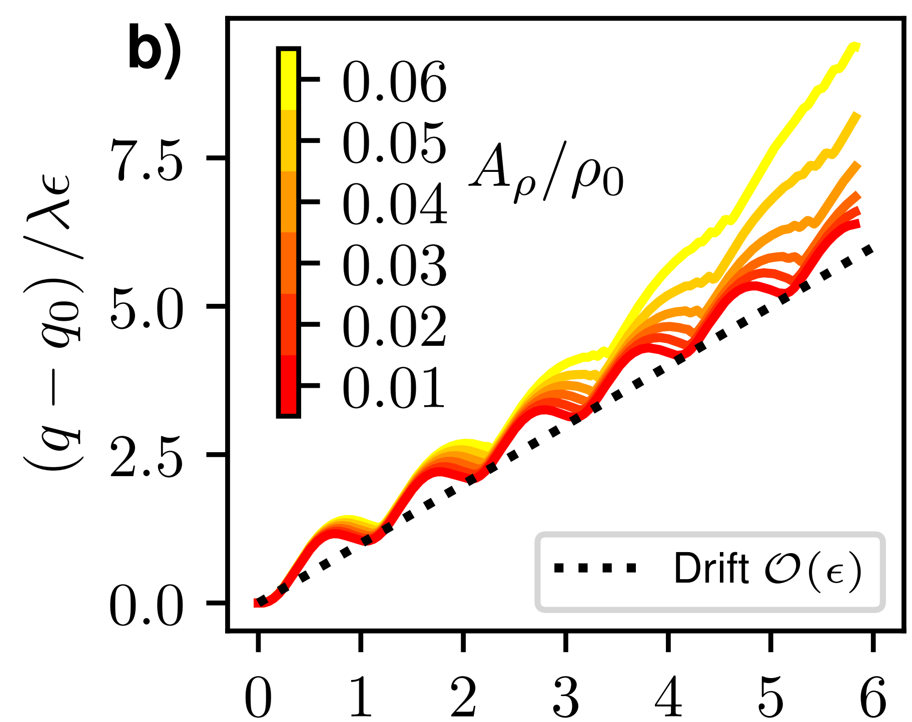


- Note that at later time nonlinear solitary waves seem to form in the quantum fluid

- This may alter the derivation of the Stokes drift which was based on sinusoidal density/phase waves

- Notice that the drift measured in the numerical simulations is enhanced when the amplitude, hence the nonlinearity, of the density/phase wave becomes larger



- Drift due to solitary waves?

CAN WE TEST OUR PREDICTIONS IN AN EXPERIMENT?

ARTICLE

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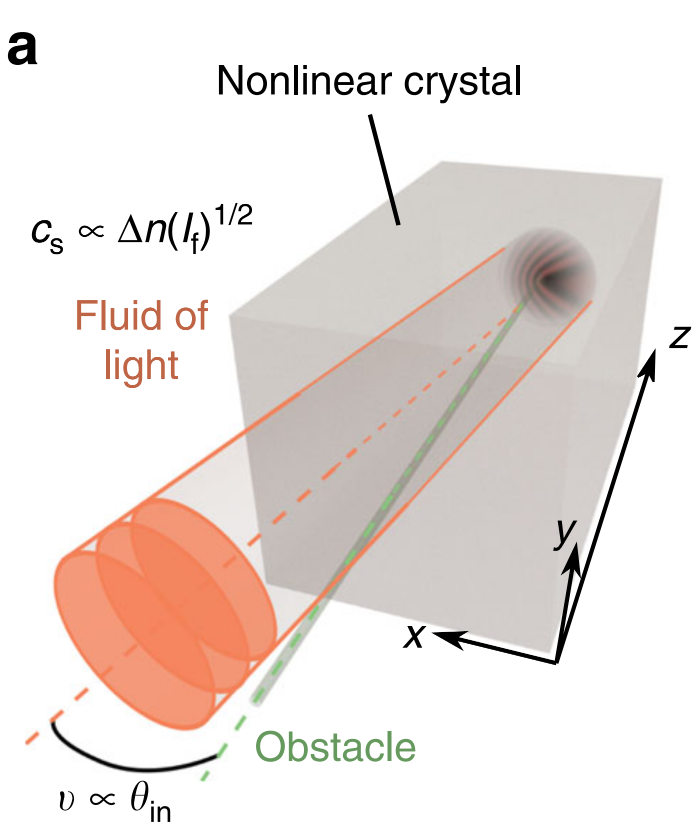
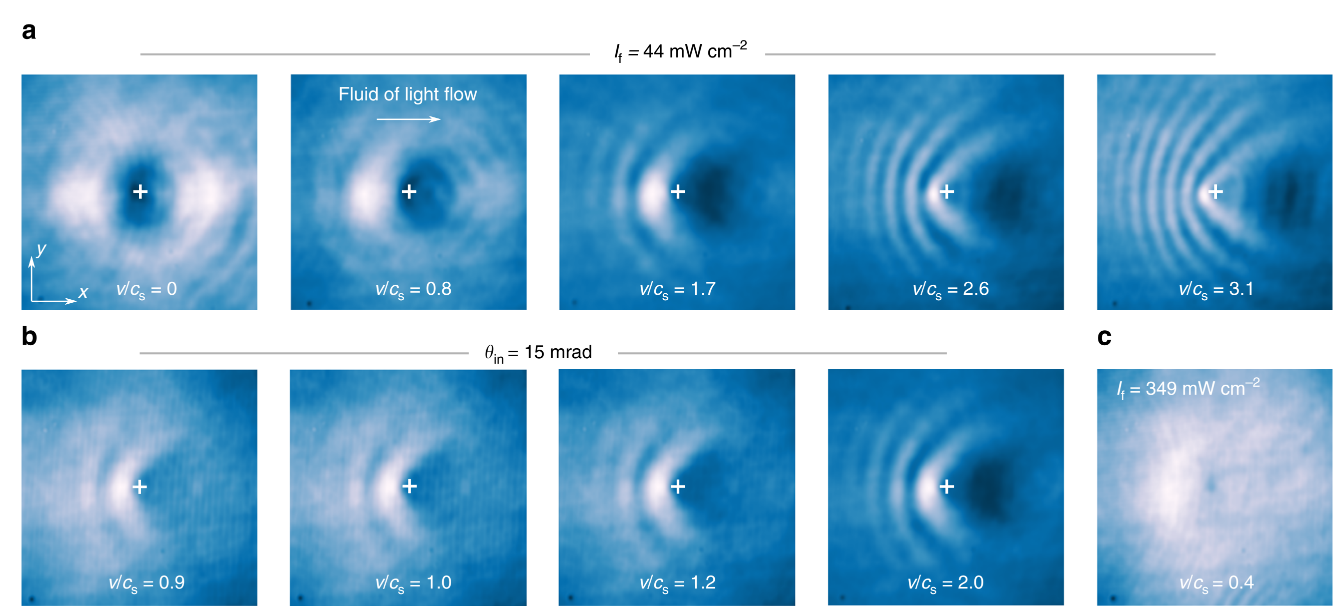
OPEN

Superfluid motion and drag-force cancellation in a fluid of light

Claire Michel[1], Omar Boughdad[1], Mathias Albert[1], Pierre-Élie Larré[2,3] & Matthieu Bellec[1]





THANKS FOR YOUR ATTENTION!

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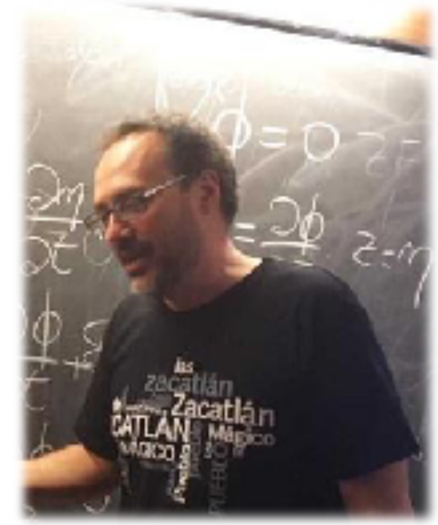

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This research was originally conceived to celebrate the bicentenary of the birth of Sir George Gabriel Stokes occurred the 13th August 1819.

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